
TECHNICAL REPORT 2006-7

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July 2006

A Class of Polynomial Volumetric Barrier Decomposition Algorithms for Stochastic Semidefinite Programming

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Abstract

Ariyawansa and Zhu [5] have recently proposed a new class of optimization problems termed stochastic semidefinite programs (SSDP's). SSDP's may be viewed as an extension of two-stage stochastic (linear) programs with recourse (SLP's). Zhao [25] has derived a decomposition algorithm for SLP's based on a logarithmic barrier and proved its polynomial complexity. Mehrotra and Özevin [17] have extended the work of Zhao [25] to the case of SSDP's to derive a polynomial logarithmic barrier decomposition algorithm for SSDP's. An alternative to the logarithmic barrier is the volumetric barrier of Vaidya [20]. It has been observed [9] that certain cutting plane algorithms [15] for SLP's based on the volumetric barrier perform better in practice than similar algorithms based on the logarithmic barrier. There is no work based on the volumetric barrier analogous to that of Zhao [25] for SLP's or to the work of Mehrotra and Özevin [17] for SSDP's. The purpose of this paper is to derive a class of volumetric barrier decomposition algorithms for SSDP's, and to prove polynomial complexity of certain members of the class.

AMS Subject Classifications: 90C15, 90C51, 49M27

Keywords: Stochastic linear programming, semidefinite programming, stochastic semidefinite programming, volumetric-barrier, self-concordance, decomposition

1 Introduction

Ariyawansa and Zhu [5] have recently proposed a new class of optimization problems termed stochastic semidefinite programs (SSDP's). SSDP's may be viewed as an extension of two-stage stochastic (linear) programs with recourse (SLP's) [7, 8, 6, 16, 19]. Alternatively, SSDP's may be viewed as an extension of (deterministic) semidefinite programs (SDP's) [1, 22, 23, 24]. See [5] for details on these relations and an application of SSDP's. Zhao [25] has derived a decomposition algorithm for SLP's based on a logarithmic barrier and proved its polynomial complexity. Mehrotra and Özevin [17] have extended the work of Zhao [25] to the case of SSDP's to derive a polynomial logarithmic barrier decomposition algorithm for SSDP's. The work of Mehrotra and Özevin [17] takes the viewpoint that SSDP's are extensions of SLP's and utilizes the work of Zhao [25].

An alternative to the logarithmic barrier is the volumetric barrier of Vaidya [20] (see also [2, 3, 4]). It has been observed [9] that certain cutting plane algorithms for SLP's based on the

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volumetric barrier perform better in practice than those based on the logarithmic barrier. The authors know of no work based on volumetric barriers analogous to that of Zhao [25] for SLP's or to the work of Mehrotra and Özevin [17] for SSDP's.

The purpose of this paper is to derive a class of decomposition algorithms for SSDP's based on a volumetric barrier, and to prove polynomial complexity of short step [3, 12] and long step [3, 12] members of the class.

While there is no work based on volumetric barriers for SLP's analogous to the work of Zhao [25], Anstreicher [4] has proved important results on volumetric barriers for SDP's. The present paper utilizes the work of Zhao [25] for SLP's, the work of Anstreicher [4] for SDP's, and the relationship of SSDP's to SLP's and SDP's described in [5], to derive volumetric barrier decomposition algorithms for SSDP's.

We begin by introducing our notation and then defining a SSDP in primal and dual standard forms. We let \mathbb{R}^+ denote the set of positive real numbers. All vectors in this paper are column vectors. We use superscript 'T' to denote transposition. The i -th unit vector is denoted by e_i . Let $\mathbb{R}^{m \times n}$ and $\mathbb{R}^{n \times n}$ denote the vector spaces of real $m \times n$ matrices and real symmetric $n \times n$ matrices respectively. For $U, V \in \mathbb{R}^{n \times n}$ we write $U \succeq 0$ ($U \succ 0$) to mean that U is positive semidefinite (positive definite), and we use $U \succeq V$ or $V \preceq U$ to mean that $U - V \succeq 0$. For $U, V \in \mathbb{R}^{m \times n}$ we write $U \bullet V := \text{trace}(U^T V)$ to denote the Frobenius inner product between U and V .

Following [5], we define a SSDP with recourse in primal standard form based on deterministic data $A_i \in \mathbb{R}^{n_1 \times n_1}$ for $i = 1, 2, \dots, m_1$, $b \in \mathbb{R}^{m_1}$ and $C \in \mathbb{R}^{n_1 \times n_1}$; and random data $T_j \in \mathbb{R}^{n_1 \times n_2}$ and $W_j \in \mathbb{R}^{n_2 \times n_2}$ for $j = 1, 2, \dots, m_2$, $h \in \mathbb{R}^{m_2}$, and $D \in \mathbb{R}^{n_2 \times n_2}$ that depend on an underlying outcome ω in an event space Ω with a known probability function P . Given this data, a SSDP with recourse in primal standard form is

$$\begin{aligned} & \text{minimize} && C \bullet X + \mathbb{E}[Q(X, \omega)] \\ & \text{subject to} && A_i \bullet X = b_i, \quad i = 1, 2, \dots, m_1 \\ & && X \succeq 0 \end{aligned} \tag{1}$$

where $X \in \mathbb{R}^{n_1 \times n_1}$ is the first-stage decision variable, $Q(X, \omega)$ is the minimum of the problem

$$\begin{aligned} & \text{minimize} && D(\omega) \bullet Y \\ & \text{subject to} && T_j(\omega) \bullet X + W_j(\omega) \bullet Y = h_j(\omega), \quad j = 1, 2, \dots, m_2 \\ & && Y \succeq 0 \end{aligned} \tag{2}$$

where $Y \in \mathbb{R}^{n_2 \times n_2}$ is the second-stage variable, and

$$\mathbb{E}[Q(X, \omega)] = \int_{\Omega} Q(X, \omega) P(d\omega). \tag{3}$$

Also following [5], we define a SSDP with recourse in dual standard form based on deterministic data $A_i \in \mathbb{R}^{n_1 \times n_1}$ for $i = 1, 2, \dots, m_1$, $b \in \mathbb{R}^{m_1}$ and $C \in \mathbb{R}^{n_1 \times n_1}$; and random data $d \in \mathbb{R}^{m_2}$, $W_i \in \mathbb{R}^{n_2 \times n_2}$ for $i = 1, 2, \dots, m_1$, $T_i \in \mathbb{R}^{n_2 \times n_2}$ for $i = 1, 2, \dots, m_2$, and $D \in \mathbb{R}^{n_2 \times n_2}$ that depend on an underlying outcome ω in an event space Ω with a known probability function P . Given this data, a SSDP with recourse in dual standard form is

$$\begin{aligned} & \text{maximize} && b^T y + \mathbb{E}[Q(y, \omega)] \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i A_i \preceq C \end{aligned} \tag{4}$$

where $y \in \mathbb{R}^{m_1}$ is the first-stage variable, $Q(y, \omega)$ is the maximum of the problem

$$\begin{aligned} & \text{maximize} && d(\omega)^\top x \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i W_i(\omega) + \sum_{i=1}^{m_2} x_i T_i(\omega) \preceq D(\omega) \end{aligned} \quad (5)$$

where $x \in \mathbb{R}^{m_2}$ is the second-stage variable, and

$$\mathbb{E}[Q(y, \omega)] = \int_{\Omega} Q(y, \omega) P(d\omega). \quad (6)$$

See [5] for a justification for referring to problem (1, 2, 3) and (4, 5, 6) as primal and dual problems respectively.

We now examine the SSDP (4, 5, 6) when the event space Ω is finite. Let $\{(d^{(k)}, (W_i^{(k)} : i = 1, 2, \dots, m_1), (T_i^{(k)} : i = 1, 2, \dots, m_2), D^{(k)}) : k = 1, 2, \dots, K\}$ be the possible values of the random variables $(d(\omega), (W_i(\omega) : i = 1, 2, \dots, m_1), (T_i(\omega) : i = 1, 2, \dots, m_2), D(\omega))$ and let $p_k := P((d(\omega), (W_i(\omega) : i = 1, 2, \dots, m_1), (T_i(\omega) : i = 1, 2, \dots, m_2), D(\omega)) = (d^{(k)}, (W_i^{(k)} : i = 1, 2, \dots, m_1), (T_i^{(k)} : i = 1, 2, \dots, m_2), D^{(k)}))$ be the associated probability for $k = 1, 2, \dots, K$. Then Problem (4, 5, 6) becomes

$$\begin{aligned} & \text{maximize} && b^\top y + \sum_{k=1}^K p_k Q^{(k)}(y) \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i A_i \preceq C, \end{aligned} \quad (7)$$

where $y \in \mathbb{R}^{m_1}$ is the first-stage variable, $Q^{(k)}(y)$ is the maximum of the problem

$$\begin{aligned} & \text{maximize} && (d^{(k)})^\top x^{(k)} \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i W_i^{(k)} + \sum_{i=1}^{m_2} x_i^{(k)} T_i^{(k)} \preceq D^{(k)}, \end{aligned} \quad (8)$$

where $x^{(k)} \in \mathbb{R}^{m_2}$ is the second-stage variable, for $k = 1, 2, \dots, K$.

In the rest of this paper our attention will be on Problem (7, 8), and from now on when we use the term stochastic semidefinite program (SSDP) in this paper we mean Problem (7, 8) or the equivalent form (9, 10) below.

The paper is organized as follows. In the next section we state some mathematical preliminaries. In §3 we introduce a volumetric barrier for the SSDP (7, 8). In §4 we show that the set of barrier functions for positive values of the barrier parameter comprises a self-concordant family [18]. Based on this property a class of volumetric barrier decomposition algorithms is presented in §5. A convergence and complexity analysis of this class of algorithms is presented in §6.

2 Preliminaries

In this section we introduce some further notation, and in order to make this paper self-contained, state some results from linear algebra and matrix calculus which we borrow from [4] (see also [11, 13, 14]).

Proposition 1. *Let $A, B \in \mathbb{R}^{n \times n}$. Then*

1. $\text{trace}(AB) = \text{trace}(BA)$;
2. if A is symmetric, then $\text{trace}(AB) = \text{trace}(AB^\top)$;
3. if A and B are positive semidefinite, then $A \bullet B \geq 0$, and $A \bullet B = 0$ if and only if $AB = 0$;
4. if $A \succeq 0$ and $B \succeq C$, then $A \bullet B \succeq A \bullet C$.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{k \times l}$ respectively. Then we define the Kronecker product $A \otimes B \in \mathbb{R}^{mk \times nl}$ of A and B as the matrix whose (i, j) block is $a_{ij}B$ for $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. We also define

$$A \otimes_s B := \frac{1}{2}(A \otimes B + B \otimes A).$$

For a matrix $A \in \mathbb{R}^{m \times n}$, we use $\text{vec}(A) \in \mathbb{R}^{mn}$ to denote the vector formed by “stacking” the columns of A one atop another in the natural order. We have

Proposition 2. *Let $A, B, C, D \in \mathbb{R}^{n \times n}$. Then*

1. $(A \otimes B)(C \otimes D) = AC \otimes BD$;
2. $(A \otimes_s B)(C \otimes_s D) = \frac{1}{2}(AC \otimes_s BD + AD \otimes_s BC)$;
3. $(A \otimes B)^\top = A^\top \otimes B^\top$;
4. if A and B are nonsingular, then $A \otimes B$ is nonsingular, and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;
5. $\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B)$;
6. if A and B are positive semidefinite, then $A \otimes B$ is positive semidefinite.

We end this section by stating the following matrix calculus results.

Proposition 3. *Let $X \in \mathbb{R}^{n \times n}$ be nonsingular, and $\det(X)$ be positive. Then*

$$\frac{\partial}{\partial x_{ij}} \ln \det(X) = [X^{-1}]_{ji},$$

and

$$\frac{\partial}{\partial x_{ij}} X^{-1} = -X^{-1} e_i e_j^\top X^{-1},$$

for $i, j = 1, 2, \dots, n$.

3 A Volumetric Barrier for SSDP’s

In this section we formulate a volumetric barrier for SSDP’s and obtain expressions for the derivatives required in the rest of the paper.

3.1 Formulation

Before we proceed, we notice that the constraints in (7, 8) are negative semidefinite while the common practice in the SDP literature is to use positive semidefinite constraints. So for convenience we rewrite the problem as follows. First we rewrite (7, 8) in the form

$$\begin{aligned} & \text{minimize} && -b^\top y + \sum_{k=1}^K p_k Q^{(k)}(y) \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i A_i - C \preceq 0, \end{aligned}$$

where for $k = 1, 2, \dots, K$, $Q^{(k)}(y)$ is the minimum of

$$\begin{aligned} & \text{minimize} && -(d^{(k)})^\top x^{(k)} \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i W_i^{(k)} + \sum_{i=1}^{m_2} x_i^{(k)} T_i^{(k)} - D^{(k)} \preceq 0. \end{aligned}$$

Then we multiply all the constraints by (-1) and replace each data vector and matrix in the problem by its negative. We get

$$\begin{aligned} & \text{minimize} && b^\top y + \sum_{k=1}^K p_k Q^{(k)}(y) \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i A_i - C \succeq 0, \end{aligned} \tag{9}$$

where for $k = 1, 2, \dots, K$, $Q^{(k)}(y)$ is the minimum of

$$\begin{aligned} & \text{minimize} && (d^{(k)})^\top x^{(k)} \\ & \text{subject to} && \sum_{i=1}^{m_1} y_i W_i^{(k)} + \sum_{i=1}^{m_2} x_i^{(k)} T_i^{(k)} - D^{(k)} \succeq 0. \end{aligned} \tag{10}$$

In order to define the volumetric barrier problem for the SSDP (9, 10), we are going to make some assumptions. First we define

$$\begin{aligned} \mathcal{F}_1 & := \left\{ y \in \mathbb{R}^{m_1} : S_1(y) := \sum_{i=1}^{m_1} y_i A_i - C \succ 0 \right\}; \\ \mathcal{F}_p^{(k)}(y) & := \left\{ x^{(k)} \in \mathbb{R}^{m_2} : S_2^{(k)}(y, x^{(k)}) := \sum_{i=1}^{m_1} y_i W_i^{(k)} + \sum_{i=1}^{m_2} x_i^{(k)} T_i^{(k)} - D^{(k)} \succ 0 \right\}; \\ \mathcal{F}_2 & := \left\{ y \in \mathbb{R}^{m_1} : \mathcal{F}_p^{(k)}(y) \neq \emptyset, k = 1, 2, \dots, K \right\}; \\ \mathcal{F}_0 & := \mathcal{F}_1 \cap \mathcal{F}_2. \end{aligned}$$

Then we make

Assumption 1. The set \mathcal{F}_0 is nonempty.

The set \mathcal{F}_1 is nonempty under Assumption 1. The logarithmic barrier [18] for \mathcal{F}_1 is the function $f_1 : \mathcal{F}_1 \rightarrow \mathbb{R}$ defined by

$$f_1(y) := -\ln \det(S_1(y)), \quad \forall y \in \mathcal{F}_1,$$

and the volumetric barrier [20, 18] for \mathcal{F}_1 is the function $V_1 : \mathcal{F}_1 \rightarrow \mathbb{R}$ defined by

$$V_1(y) := \frac{1}{2} \ln \det(\nabla^2 f_1(y)), \quad \forall y \in \mathcal{F}_1.$$

Also under Assumption 1, \mathcal{F}_2 is nonempty and for $y \in \mathcal{F}_2$, $\mathcal{F}_p^{(k)}(y)$ is nonempty for $k = 1, 2, \dots, K$. The logarithmic barrier [18] for $\mathcal{F}_p^{(k)}(y)$ is the function $f_2^{(k)} : \mathcal{F}_p^{(k)}(y) \rightarrow \mathbb{R}$ defined by

$$f_2^{(k)}(y, x^{(k)}) := -\ln \det(S_2^{(k)}(y, x^{(k)})), \quad \forall x^{(k)} \in \mathcal{F}_p^{(k)}(y), y \in \mathcal{F}_2,$$

and the volumetric barrier [20, 18] for $\mathcal{F}_p^{(k)}(y)$ is the function $V_2^{(k)} : \mathcal{F}_p^{(k)}(y) \rightarrow \mathbb{R}$ defined by

$$V_2^{(k)}(y, x^{(k)}) := \frac{1}{2} \ln \det(\nabla_{x^{(k)} x^{(k)}}^2 f_2^{(k)}(y, x^{(k)})), \quad \forall x^{(k)} \in \mathcal{F}_p^{(k)}(y), y \in \mathcal{F}_2.$$

Next we make

Assumption 2. For each $y \in \mathcal{F}_0$ and for $k = 1, 2, \dots, K$, Problem (10) has a nonempty isolated compact set of minimizers.

We now define the volumetric barrier problem for the SSDP (9, 10) as

$$\text{minimize } \eta(\mu, y) := b^\top y + \sum_{k=1}^K p_k \rho_k(\mu, y) + \mu c_1 V_1(y), \quad (11)$$

where for $k = 1, 2, \dots, K$ and $y \in \mathcal{F}_0$, $\rho_k(\mu, y)$ is the minimum of

$$\text{minimize } (d^{(k)})^\top x^{(k)} + \mu c_2 V_2^{(k)}(y, x^{(k)}). \quad (12)$$

Here $c_1 > 0$ and $c_2 > 0$ are constants whose values will be defined more precisely in the sequel and $\mu > 0$ is the barrier parameter.

We will now show that (12) has a unique minimizer for each $y \in \mathcal{F}_0$ and for $k = 1, 2, \dots, K$ by utilizing:

Theorem 1 (Fiacco and McCormick [10, Theorem 8]). *Consider the inequality constrained problem*

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } g_i(x) \geq 0, \quad i = 1, 2, \dots, m, \end{aligned} \quad (13)$$

where the functions $f, g_1, \dots, g_m : \mathbb{R}^n \rightarrow \mathbb{R}$ are continuous. Let I be a scalar-valued function of x with the following two properties: $I(x)$ is continuous in the region $R^0 := \{x : g_i(x) > 0, i =$

$1, 2, \dots, m\}$, which is assumed to be nonempty; if $\{x_k\}$ is any infinite sequence of points in R^0 converging to x_B such that $g_i(x_B) = 0$ for at least one i , then $\lim_{k \rightarrow \infty} I(x_k) = +\infty$. Let s be a scalar-valued function of the single variable r with the following two properties: if $r_1 > r_2 > 0$, then $s(r_1) > s(r_2) > 0$; if $\{r_k\}$ is an infinite sequence of points such that $\lim_{k \rightarrow \infty} r_k = 0$, then $\lim_{k \rightarrow \infty} s(r_k) = 0$. Let $U : R^0 \times \mathbb{R}^+ \rightarrow \mathbb{R}$ be defined by $U(x, r) := f(x) + s(r)I(x)$. If (13) has a nonempty, isolated compact set of local minimizers and $\{r_k\}$ is a strictly decreasing infinite sequence, then the unconstrained local minimizers of $U(\cdot, r_k)$ exist for r_k small.

Lemma 1. *If Assumptions 1 and 2 hold, then for each $y \in \mathcal{F}_0$ and $k = 1, 2, \dots, K$, the Problem (12) has a unique minimizer for μ small.*

Proof. For any given $y \in \mathcal{F}_0$, $V_2^{(k)}(y, x^{(k)})$ is defined on the nonempty set $\mathcal{F}_p^{(k)}(y)$. The positive definite matrix $S_2^{(k)}(y, x^{(k)})$ can be factored into the product of three matrices: a unit lower triangular matrix L , a positive definite diagonal matrix D , and the transpose of L , such that $S_2^{(k)}(y, x^{(k)}) = LDL^\top$. Let d_j denote the j -th diagonal element of D viewed as a function of $x^{(k)} \in \mathcal{F}_p^{(k)}(y)$, for $j = 1, 2, \dots, n_2$. Then d_j is continuous for $j = 1, 2, \dots, n_2$ [21]. Then the constraint $S_2^{(k)}(y, x^{(k)}) \succ 0$ can be replaced by the constraints: $d_j(x^{(k)}) > 0$, $j = 1, 2, \dots, n_2$ [21]. So (10) can be rewritten in the form of (13). Therefore, by Theorem 1, local minimizers of (12) exist for each $y \in \mathcal{F}_0$ and $k = 1, 2, \dots, K$ for μ small. The uniqueness of the minimizer follows from the fact that $V_2^{(k)}$ is strictly convex. \square

By Lemma 1, the Problem (11) is well-defined, and its feasible set is \mathcal{F}_0 .

3.2 Expressions for partial derivatives of η with respect to y

In order to compute the derivatives of η we need the derivatives of ρ_k , $k = 1, 2, \dots, k$, which in turn require the derivatives of $V_2^{(k)}$ and $f_2^{(k)}$ for each $k = 1, 2, \dots, K$. Some of these computations are lengthy and it is convenient to drop the superscript (k) . We do so when it does not lead to confusion.

Also we make

Assumption 3. Matrices $T_i^{(k)}$, $i = 1, 2, \dots, m_2$ are linear independent for $k = 1, 2, \dots, K$.

Let $\mathcal{T} \in \mathbb{R}^{n_1^2 \times m_2}$ be the matrix whose i -th column is $\text{vec}(T_i) \in \mathbb{R}^{n_1^2}$ for $i = 1, 2, \dots, m_2$. Then the Hessian matrix $H := \nabla_{xx}^2 f_2(y, x)$ can be represented in the form (See also [1].)

$$H := \nabla_{xx}^2 f_2(y, x) = \mathcal{T}^\top [S_2^{-1} \otimes S_2^{-1}] \mathcal{T}.$$

Note that by Proposition 2 and Assumption 3, H is positive definite. We have (See also [4].):

$$\begin{aligned} \frac{\partial V_2(y, x)}{\partial x_i} &= -(\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_iS_2^{-1} \otimes_s S_2^{-1}) \\ &= -P \bullet (S_2^{-1/2}T_iS_2^{-1/2} \otimes_s I), \end{aligned}$$

for $i = 1, 2, \dots, m_2$, and

$$\begin{aligned} \frac{\partial V_2(y, x)}{\partial y_i} &= -(\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_iS_2^{-1} \otimes_s S_2^{-1}) \\ &= -P \bullet (S_2^{-1/2}W_iS_2^{-1/2} \otimes_s I), \end{aligned} \tag{14}$$

for $i = 1, 2, \dots, m_1$, where

$$P = P(S_2) = (S_2^{-1/2} \otimes S_2^{-1/2})\mathcal{T}(\mathcal{T}^\top(S_2^{-1} \otimes S_2^{-1})\mathcal{T})^{-1}\mathcal{T}^\top(S_2^{-1/2} \otimes S_2^{-1/2})$$

is the orthogonal projection onto the range of $(S_2^{-1/2} \otimes S_2^{-1/2})\mathcal{T}$;

$$\nabla_{xy}^2 V_2(y, x) = \frac{\partial^2}{\partial y \partial x} V_2(y, x) = 2Q^{xy} + R^{xy} - 2T^{xy},$$

where

$$\begin{aligned} Q_{i,j}^{xy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1}W_j S_2^{-1} \otimes_s S_2^{-1}), \\ R_{i,j}^{xy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1} \otimes_s S_2^{-1}W_j S_2^{-1}), \\ T_{i,j}^{xy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1} \otimes_s S_2^{-1})\mathcal{T}H^{-1}\mathcal{T}^\top(S_2^{-1}W_j S_2^{-1} \otimes_s S_2^{-1}); \end{aligned}$$

$$\nabla_{yx}^2 V_2(y, x) = \frac{\partial^2}{\partial x \partial y} V_2(y, x) = 2Q^{yx} + R^{yx} - 2T^{yx},$$

where

$$\begin{aligned} Q_{i,j}^{yx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1}), \\ R_{i,j}^{yx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1}T_j S_2^{-1}), \\ T_{i,j}^{yx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1})\mathcal{T}H^{-1}\mathcal{T}^\top(S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1}); \end{aligned}$$

$$\nabla_{xx}^2 V_2(y, x) = \frac{\partial^2}{\partial x \partial x} V_2(y, x) = 2Q^{xx} + R^{xx} - 2T^{xx},$$

where

$$\begin{aligned} Q_{i,j}^{xx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1}), \\ R_{i,j}^{xx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1} \otimes_s S_2^{-1}T_j S_2^{-1}), \\ T_{i,j}^{xx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_i S_2^{-1} \otimes_s S_2^{-1})\mathcal{T}H^{-1}\mathcal{T}^\top(S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1}); \end{aligned}$$

$$\nabla_{yy}^2 V_2(y, x) = \frac{\partial^2}{\partial y \partial y} V_2(y, x) = 2Q^{yy} + R^{yy} - 2T^{yy}, \quad (15)$$

where

$$\begin{aligned} Q_{i,j}^{yy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1}W_j S_2^{-1} \otimes_s S_2^{-1}), \\ R_{i,j}^{yy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1}W_j S_2^{-1}), \\ T_{i,j}^{yy} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1})\mathcal{T}H^{-1}\mathcal{T}^\top(S_2^{-1}W_j S_2^{-1} \otimes_s S_2^{-1}). \end{aligned}$$

Proposition 4.

$$\nabla_{xy}^2 V_2(y, x) = [\nabla_{yx}^2 V_2(y, x)]^\top. \quad (16)$$

Proof. First we show $Q_{i,j}^{yx} = Q_{j,i}^{xy}$. We have

$$\begin{aligned} Q_{i,j}^{yx} &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1}) \\ &= \text{trace}((\mathcal{T}H^{-1}\mathcal{T}^\top)(S_2^{-1}W_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1})^\top) \\ &= \text{trace}((\mathcal{T}H^{-1}\mathcal{T}^\top)(S_2^{-1}W_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1})) \\ &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}W_i S_2^{-1}T_j S_2^{-1} \otimes_s S_2^{-1})^\top \\ &= (\mathcal{T}H^{-1}\mathcal{T}^\top) \bullet (S_2^{-1}T_j S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1}) \\ &= Q_{j,i}^{xy}. \end{aligned}$$

By similar arguments we can show that $R_{i,j}^{yx} = R_{j,i}^{xy}$ and $T_{i,j}^{yx} = T_{j,i}^{xy}$. Hence

$$\nabla_{xy}^2 V_2(y, x) = [\nabla_{yx}^2 V_2(y, x)]^\top. \quad \square$$

Proposition 5. Let (y, x) be such that $S_2(y, x) \succ 0$. Then we have

$$0 \preceq Q^{yy} \preceq \nabla_{yy}^2 V_2(y, x). \quad (17)$$

Proof. (See also [4].) Let $\xi \in \mathbb{R}^{m_1}$, $\xi \neq 0$. We have

$$\xi^\top Q^{yy} \xi = \sum_{i,j} Q_{ij}^{yy} \xi_i \xi_j = \mathcal{T}H^{-1}\mathcal{T} \bullet (S_2^{-1}BS_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}) = P \bullet (\bar{B}^2 \otimes_s I),$$

where $B := B(\xi) := \sum_{i=1}^{m_1} \xi_i W_i$, and $\bar{B} := S_2^{-1/2}BS_2^{-1/2}$. Similarly we have

$$\xi^\top R^{yy} \xi = \mathcal{T}H^{-1}\mathcal{T} \bullet (S_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}BS_2^{-1}) = P \bullet (\bar{B} \otimes \bar{B}),$$

and

$$\begin{aligned} \xi^\top T^{yy} \xi &= \mathcal{T}H^{-1}\mathcal{T} \bullet (S_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}) \mathcal{T}H^{-1}\mathcal{T} (S_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}) \\ &= P \bullet (\bar{B} \otimes_s I) P (\bar{B} \otimes_s I). \end{aligned}$$

Since I , P and \bar{B}^2 are all positive semidefinite, we immediately have $Q^{yy} \succeq 0$ from Proposition 1 and Proposition 2. In addition, P is a projection implies that

$$(\bar{B} \otimes_s I) P (\bar{B} \otimes_s I) \preceq (\bar{B} \otimes_s I) (\bar{B} \otimes_s I) = \frac{1}{2} ((\bar{B}^2 \otimes_s I) + (\bar{B} \otimes \bar{B})). \quad (18)$$

We conclude that

$$P \bullet (\bar{B} \otimes_s I) P (\bar{B} \otimes_s I) \leq \frac{1}{2} P \bullet ((\bar{B}^2 \otimes_s I) + (\bar{B} \otimes \bar{B})),$$

which is exactly $\xi^\top T^{yy} \xi \leq \frac{1}{2} \xi^\top (Q^{yy} + R^{yy}) \xi$. Since ξ is arbitrary, we have shown that $T^{yy} \preceq \frac{1}{2} (Q^{yy} + R^{yy})$, which together with $Q^{yy} \succeq 0$ implies

$$0 \preceq Q^{yy} \preceq \nabla_{yy}^2 V_2(y, x). \quad \square$$

Proposition 6. For any $\xi \in \mathbb{R}^{m_1}$, let $\bar{B} := S_2^{-1/2}(\sum_{i=1}^{m_1} \xi_i W_i)S_2^{-1/2}$. Let (y, x) be such that $S_2(y, x) \succ 0$. Then

$$\xi^\top Q^{yy} \xi \geq \frac{1}{n_2^3} |\bar{B}|^2, \quad (19)$$

$$i.e. |\bar{B}| \leq n_2^{3/2} (\xi^\top Q^{yy} \xi)^{1/2}.$$

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_{n_2}$ be the eigenvalues of \bar{B} , with corresponding orthonormal eigenvectors v_1, v_2, \dots, v_{n_2} . Then $\bar{B}^2 \otimes_s I$ has a full set of orthonormal eigenvectors $v_i \otimes v_j$, with corresponding eigenvalues $(1/2)(\lambda_i^2 + \lambda_j^2)$ for $i, j = 1, 2, \dots, n_2$. We have

$$\begin{aligned} \xi^\top Q \xi &= P \cdot \frac{1}{2} \sum_{i,j=1}^{n_2} (\lambda_i^2 + \lambda_j^2) (v_i \otimes v_j) (v_i \otimes v_j)^\top \\ &= \frac{1}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) (v_i \otimes v_j)^\top P (v_i \otimes v_j). \end{aligned}$$

Let $\mu_1, \mu_2, \dots, \mu_{n_2^2}$ be the eigenvalues of P with corresponding orthonormal eigenvectors $u_1, u_2, \dots, u_{n_2^2}$. Then

$$P = \sum_{l=1}^{n_2^2} \mu_l u_l u_l^\top.$$

Since P is a projection from an $n_2^2 \times n_2^2$ space onto an n_2^2 space, we have $\mu_1 = \mu_2 = \dots = \mu_{n_2} = 1$ (rearrange indices if necessary). Consider u_k for some k , we have

$$u_k = \sum_{i,j=1}^{n_2} c_{ij} (v_i \otimes v_j),$$

for some constants c_{ij} , for $i, j = 1, 2, \dots, n_2$, and

$$\begin{aligned} 1 &= \|u_k\|_2 = \left\| \sum_{i,j=1}^{n_2} c_{ij} (v_i \otimes v_j) \right\|_2 \\ &\leq \sum_{i,j=1}^{n_2} \|c_{ij} (v_i \otimes v_j)\|_2 \\ &= \sum_{i,j=1}^{n_2} |c_{ij}|. \end{aligned}$$

Thus there exist i_k, j_k such that

$$|u_k^\top (v_{i_k} \otimes v_{j_k})| = |c_{i_k j_k}| \geq \frac{1}{n_2^2}.$$

Hence

$$\begin{aligned} \xi^\top Q \xi &= \frac{1}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) (v_i \otimes v_j)^\top \left(\sum_{l=1}^{n_2^2} \mu_l u_l u_l^\top \right) (v_i \otimes v_j) \\ &= \frac{1}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) \sum_{l=1}^{n_2^2} \mu_l ((v_i \otimes v_j)^\top u_l) (u_l^\top (v_i \otimes v_j)) \\ &= \frac{1}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) \sum_{l=1}^{n_2^2} \mu_l |u_l^\top (v_i \otimes v_j)|^2 \\ &= \frac{1}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) \sum_{k=1}^{n_2} |u_k^\top (v_i \otimes v_j)|^2 \\ &\geq \frac{1}{2} \sum_{k=1}^{n_2} ((\lambda_{i_k}^2 + \lambda_{j_k}^2) |u_k^\top (v_{i_k} \otimes v_{j_k})|^2) \\ &\geq \frac{1}{2n_2^3} (\lambda_{i_k}^2 + \lambda_{j_k}^2) \geq \frac{1}{n_2^3} (\lambda_{i_k} \lambda_{j_k}) \geq \frac{1}{n_2^3} (\lambda_{i_k}^2 \lambda_{j_k}^2) \\ &\geq \frac{1}{n_2^3} \prod_{i=1}^{n_2} (\lambda_i^2) = \frac{1}{n_2^3} |\bar{B}|^2. \quad \square \end{aligned}$$

Now for the Problem (12) we have

$$\rho_k(\mu, y) = d^\top \bar{x} + \mu c_2 V_2(y, \bar{x}), \quad (20)$$

where \bar{x} is the optimal solution for (12). We notice that \bar{x} is a function of y and the relationship between \bar{x} and y is defined by

$$\frac{\partial}{\partial x}[d^\top x + \mu c_2 V_2(y, x)]|_{x=\bar{x}} = 0. \quad (21)$$

Lemma 2. *Let f be a differentiable function of x and y . Suppose that $x := \bar{x}(y)$ satisfies*

$$f_x(x, y) = 0. \quad (22)$$

Then

$$f_{\bar{x}(y)}(\bar{x}, y) = 0. \quad (23)$$

Proof. $f_x(x, y)|_{x=\bar{x}} = 0$ since \bar{x} satisfies (23). Noting that

$$f_{\bar{x}}(\bar{x}, y) = f_x(x, y)|_{x=\bar{x}},$$

the result follows. □

Proposition 7.

$$\frac{\partial}{\partial \bar{x}}[d^\top \bar{x} + \mu c_2 V_2(y, \bar{x})] = 0. \quad (24)$$

Proof. By (21), \bar{x} satisfies

$$\frac{\partial}{\partial x}[d^\top x + \mu c_2 V_2(y, x)] = 0.$$

The result then follows from Lemma 2. □

Proposition 8.

$$\frac{\partial^2}{\partial y \partial \bar{x}} V_2(y, \bar{x}) = 0. \quad (25)$$

Proof. By (24)

$$\frac{\partial}{\partial \bar{x}} V_2(y, \bar{x}) = -d^{(k)} / (\mu c_2).$$

The result follows from the fact that $d^{(k)}$, μ and c_2 are not functions of y . □

Now we are ready to calculate the first and second order derivatives of ρ_k with respect to y . We have

$$\begin{aligned} \nabla_y \rho_k(\mu, y) &= \nabla_y (d^\top \bar{x} + \mu c_2 V_2(y, \bar{x})) \\ &= \frac{\partial \rho_k(\mu, y)}{\partial \bar{x}} \cdot \frac{\partial \bar{x}}{\partial y} + \frac{\partial \rho_k(\mu, y)}{\partial y} \\ &= \frac{\partial \rho_k(\mu, y)}{\partial y} \quad (\text{by (24)}) \\ &= \mu c_2 \nabla_y V_2(y, \bar{x}), \end{aligned}$$

and

$$\begin{aligned} \nabla_{yy}^2 \rho_k(\mu, y) &= \nabla_y (\nabla_y \rho_k(\mu, y)) \\ &= \nabla_y (\mu c_2 \nabla_y V_2(y, \bar{x})) \\ &= \mu c_2 \frac{\partial}{\partial \bar{x}} (\nabla_y V_2(y, \bar{x})) \cdot \frac{\partial \bar{x}}{\partial y} + \mu c_2 \frac{\partial}{\partial y} (\nabla_y V_2(y, \bar{x})) \\ &= \mu c_2 \frac{\partial^2}{\partial \bar{x} \partial y} V_2(y, \bar{x}) \cdot \frac{\partial \bar{x}}{\partial y} + \mu c_2 \frac{\partial^2}{\partial y \partial y} V_2(y, \bar{x}) \\ &= \mu c_2 \nabla_{yy}^2 V_2(y, \bar{x}) \quad (\text{by (25)}). \end{aligned}$$

Theorem 2 (Clairaut's Theorem). Suppose f is a function of two variables and it's defined on a disk D that contains the point (a, b) . Let

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x},$$

and

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}.$$

If the functions f_{xy} and f_{yx} are both continuous on this disk, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Theorem 3 (An extension of Clairaut's Theorem). Let

$$f_{xxy} = \frac{\partial^3 f}{\partial y \partial x \partial x},$$

$$f_{xyx} = \frac{\partial^3 f}{\partial x \partial y \partial x},$$

and

$$f_{yxx} = \frac{\partial^3 f}{\partial x \partial x \partial y}.$$

If the functions f_{xyx} , f_{yxx} and f_{xxy} are all continuous, then

$$f_{xxy} = f_{xyx} = f_{yxx}. \quad (26)$$

Proof. If the third order derivatives are continuous then the second order ones are also continuous. So by Clairaut's Theorem we have

$$f_{xxy} = (f_x)_{xy} = (f_x)_{yx} = f_{xyx} = (f_{xy})_x = (f_{yx})_x = f_{yxx}. \quad \square$$

In general, we can extend Clairaut's Theorem to any order of mixed partial derivatives. The only requirement is that in each derivative we differentiate with respect to each variable the same number of times.

Now we have

$$\begin{aligned} \nabla_{yyy}^3 \rho_k(\mu, y) &= \nabla_y(\nabla_{yy}^2 \rho_k(\mu, y)) \\ &= \nabla_y(\mu c_2 \nabla_{yy}^2 V_2(y, \bar{x})) \\ &= \mu c_2 \frac{\partial}{\partial \bar{x}}(\nabla_{yy}^2 V_2(y, \bar{x})) \cdot \frac{\partial \bar{x}}{\partial y} + \mu c_2 \frac{\partial}{\partial y}(\nabla_{yy}^2 V_2(y, \bar{x})) \\ &= \mu c_2 \frac{\partial^3}{\partial \bar{x} \partial y \partial y} V_2(y, \bar{x}) \cdot \frac{\partial \bar{x}}{\partial y} + \mu c_2 \frac{\partial^3}{\partial y \partial y \partial y} V_2(y, \bar{x}) \\ &= \mu c_2 \frac{\partial^3}{\partial y \partial y \partial \bar{x}} V_2(y, \bar{x}) \cdot \frac{\partial \bar{x}}{\partial y} + \mu c_2 \frac{\partial^3}{\partial y \partial y \partial y} V_2(y, \bar{x}) \\ &= \mu c_2 \nabla_{yyy}^3 V_2(y, \bar{x}) \quad (\text{by (25) and (26)}). \end{aligned}$$

In summary we have

$$\begin{aligned} \nabla_y \rho_k(\mu, y) &= \mu c_2 \nabla_y V_2^{(k)}(y, \bar{x}^{(k)}), \\ \nabla_{yy}^2 \rho_k(\mu, y) &= \mu c_2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)}), \\ \nabla_{yyy}^3 \rho_k(\mu, y) &= \mu c_2 \nabla_{yyy}^3 V_2^{(k)}(y, \bar{x}^{(k)}), \end{aligned} \quad (27)$$

and

$$\begin{aligned}\nabla_y \eta(\mu, y) &= b + \mu c_1 \nabla_y V_1(y) + \sum_{k=1}^K \mu c_2 \nabla_y V_2^{(k)}(y, \bar{x}^{(k)}), \\ \nabla_{yy}^2 \eta(\mu, y) &= \mu c_1 \nabla_y^2 V_1(y) + \sum_{k=1}^K \mu c_2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)}),\end{aligned}\tag{28}$$

where $\nabla_y V_2^{(k)}(y, \bar{x}^{(k)})$, $\nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)})$, and $\nabla_{yyy}^3 V_2^{(k)}(y, \bar{x}^{(k)})$ are as in (14), (15) and (29) respectively.

4 Characteristics of η : a self-concordant family

4.1 Self-Concordance of $\eta(\mu, \cdot)$

Definition 1 (Nesterov and Nemirovskii [18, Definition 2.1.1]). Let G be an open nonempty convex subset of \mathbb{R}^n , and let F be a \mathcal{C}^3 , convex mapping from G to \mathbb{R} . Then F is called α -self-concordant on G with the parameter $\alpha > 0$ if for every $x \in G$ and $\xi \in \mathbb{R}^n$, the following inequality holds:

$$|D^3 F(x)[\xi, \xi, \xi]| \leq 2\alpha^{-1/2} (D^2 F(x)[\xi, \xi])^{3/2}.$$

An α -self-concordant function F on G is called strongly α -self-concordant if F tends to infinity for any sequence approaching a boundary point of G .

We note that in the definition above the set G is assumed to be open. However, relative openness would be sufficient to apply the definition. See also [18, Item A, Page 57]. We now show that $\rho_k(\mu, \cdot)$ is μ -self-concordant on \mathcal{F}_0 , for $k = 1, 2, \dots, K$. It is clear that \mathcal{F}_0 is open.

Theorem 4. For any fixed $\mu > 0$ and $c_2 := 225 n_2^{3/2}$, $\rho_k(\mu, \cdot)$ is μ -self-concordant on \mathcal{F}_0 , for $k = 1, 2, \dots, K$.

In order to prove Theorem 4 we need some intermediate results which we now obtain. Let (y, x) be such that $S_2(y, x) \succ 0$, and $\xi \in \mathbb{R}^{m_1}$. We immediately obtain

$$\begin{aligned}\frac{\partial}{\partial y_i} \xi^\top Q^{yy} \xi &= 2TH^{-1}T^\top (S_2^{-1}W_i S_2^{-1} \otimes_s S_2^{-1}) TH^{-1}T^\top \\ &\quad \cdot (S_2^{-1}BS_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}) \\ &\quad + TH^{-1}T \cdot \frac{\partial}{\partial y_i} (S_2^{-1}BS_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}),\end{aligned}$$

where $B := B(\xi) := \sum_{i=1}^{m_1} \xi_i W_i$, and

$$\begin{aligned}&\frac{\partial}{\partial y_i} (S_2^{-1}BS_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}) \\ &= -(S_2^{-1}W_i S_2^{-1}BS_2^{-1}BS_2^{-1} + S_2^{-1}BS_2^{-1}W_i S_2^{-1}BS_2^{-1} + S_2^{-1}BS_2^{-1}BS_2^{-1}W_i S_2^{-1}) \\ &\quad \otimes_s S_2^{-1} - S_2^{-1}BS_2^{-1}BS_2^{-1} \otimes_s S_2^{-1}W_i S_2^{-1}.\end{aligned}$$

We conclude that the first directional derivative of $\xi^\top Q^{yy} \xi$ with respect to y , in the direction ξ , is given by

$$\begin{aligned}\nabla_y \xi^\top Q^{yy} \xi [\xi] &= \sum_{i=1}^{m_1} \xi_i \frac{\partial}{\partial y_i} \xi^\top Q^{yy} \xi \\ &= 2P \cdot (\bar{B} \otimes_s I) P (\bar{B}^2 \otimes_s I) - 3P \cdot (\bar{B}^3 \otimes_s I) - P \cdot (\bar{B}^2 \otimes_s \bar{B}),\end{aligned}$$

where $\bar{B} := S^{-1/2}BS^{-1/2}$, and P is defined as before. Similarly we obtain

$$\begin{aligned}\nabla_y \xi^\top R^{yy} \xi [\xi] &= 2P \cdot (\bar{B} \otimes_s I)P(\bar{B} \otimes_s \bar{B}) - 4P \cdot (\bar{B}^2 \otimes_s \bar{B}), \\ \nabla_y \xi^\top T^{yy} \xi [\xi] &= 4P \cdot (\bar{B} \otimes_s I)P(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I) - 4P \cdot (\bar{B} \otimes_s I)P(\bar{B}^2 \otimes_s I) \\ &\quad - 2P \cdot (\bar{B} \otimes_s I)P(\bar{B} \otimes_s \bar{B}).\end{aligned}$$

Combining the previous results, we obtain the third order directional derivative of $V_2(y, x)$ with respect to y as:

$$\begin{aligned}\nabla_{yyy}^3 V_2(y, x) [\xi, \xi, \xi] &= 12P \cdot (\bar{B} \otimes_s I)P(\bar{B}^2 \otimes_s I) \\ &\quad - 6P \cdot (\bar{B}^3 \otimes_s I) - 6P \cdot (\bar{B}^2 \otimes_s \bar{B}) \\ &\quad + 6P \cdot (\bar{B} \otimes_s I)P(\bar{B} \otimes_s \bar{B}) \\ &\quad - 8P \cdot (\bar{B} \otimes_s I)P(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I).\end{aligned}\tag{29}$$

In the proof of Theorem 4 we need to bound $\nabla_{yyy}^3 V_2(y, x) [\xi, \xi, \xi]$. We now obtain such a bound.

Proposition 9. *For any $\xi \in \mathbb{R}^{m_1}$, let $\bar{B} := S_2^{-1/2}(\sum_{i=1}^{m_1} \xi_i W_i)S_2^{-1/2}$. Let (y, x) be such that $S_2(y, x) \succ 0$. Then*

$$|\nabla_{yyy}^3 V_2(y, x) [\xi, \xi, \xi]| \leq 30|\bar{B}|\xi^\top Q^{yy} \xi.\tag{30}$$

Proof. (See also [4].) Using the fact that

$$(\bar{B}^2 \otimes_s I)(\bar{B} \otimes_s I) = \frac{1}{2}((\bar{B}^3 \otimes_s I) + (\bar{B}^2 \otimes_s \bar{B})),$$

(29) can be re-written as

$$\begin{aligned}\nabla_{yyy}^3 V_2(y, x) [\xi, \xi, \xi] &= P(\bar{B} \otimes_s I)P \cdot (12(\bar{B}^2 \otimes_s I) \\ &\quad + 6(\bar{B} \otimes_s \bar{B}) - 8(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I) \\ &\quad - 12P \cdot (\bar{B}^2 \otimes_s I)(\bar{B} \otimes_s I)).\end{aligned}\tag{31}$$

From (18) we have

$$12(\bar{B}^2 \otimes_s I) + 6(\bar{B} \otimes_s \bar{B}) - 8(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I) \succeq 8(\bar{B}^2 \otimes_s I) + 2(\bar{B} \otimes_s \bar{B}).$$

Using the facts that $(\bar{B}^2 \otimes_s I) \succeq (\bar{B} \otimes_s \bar{B})$ and $(\bar{B} \otimes_s \bar{B}) \succeq (-\bar{B}^2 \otimes_s I)$, it follows that

$$\begin{aligned}6(\bar{B}^2 \otimes_s I) &\leq 12(\bar{B}^2 \otimes_s I) + 6(\bar{B} \otimes_s \bar{B}) - 8(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I) \\ &\leq 18(\bar{B}^2 \otimes_s I).\end{aligned}\tag{32}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{m_1}$ be the eigenvalues of \bar{B} . Then the eigenvalues of $(\bar{B} \otimes_s I)$ are of the form $(1/2)(\lambda_i + \lambda_j)$, for $i, j = 1, 2, \dots, m_1$. We have

$$-|\bar{B}|I \preceq (\bar{B} \otimes_s I) \preceq |\bar{B}|I,$$

and

$$-|\bar{B}|P \preceq P(\bar{B} \otimes_s I)P \preceq |\bar{B}|P.\tag{33}$$

Using (32), (33), and the face that $(\bar{B}^2 \otimes_s I) \succeq 0$, we obtain

$$\begin{aligned}&|P(\bar{B} \otimes_s I) \cdot (12(\bar{B}^2 \otimes_s I) + 6(\bar{B} \otimes_s \bar{B}) - 8(\bar{B} \otimes_s I)P(\bar{B} \otimes_s I))| \\ &\leq 12|\bar{B}|P \cdot (\bar{B}^2 \otimes_s I).\end{aligned}\tag{34}$$

In addition, $(\bar{B}^2 \otimes_s I)$ and $(\bar{B} \otimes_s I)$ have the same eigenvectors implies that

$$-|\bar{B}|(\bar{B}^2 \otimes_s I) \preceq (\bar{B}^2 \otimes_s I)(\bar{B} \otimes_s I) \preceq |\bar{B}|(\bar{B}^2 \otimes_s I).$$

Therefore we have

$$|P \cdot (\bar{B}^2 \otimes_s I)(\bar{B} \otimes_s I)| \leq |\bar{B}|P \cdot (\bar{B}^2 \otimes_s I). \quad (35)$$

The conclusion follows from (31), (34) and (35). \square

We can now state the proof of Theorem 4.

Proof of Theorem 4. Combining the results of (17), (19) and (30), we obtain

$$|\nabla_{yyy}^3 V_2(y, \bar{x}) [\xi, \xi, \xi]| \leq 30n_2^{3/2} (\xi^\top \nabla_{yy}^2 V_2(y, \bar{x}) \xi)^{3/2},$$

which combined with (27) gives us

$$\begin{aligned} |\nabla_{yyy}^3 \rho_k(y) [\xi, \xi, \xi]| &\leq 30\mu c_2 n_2^{3/2} (\nabla_{yy}^2 V_2(y, \bar{x}) [\xi, \xi])^{3/2} \\ &= 2\mu^{-1/2} (c_2 \mu \nabla_{yy}^2 V_2(y, x) [\xi, \xi])^{3/2} \\ &= 2\mu^{-1/2} (\nabla_{yy}^2 \rho_k(y) [\xi, \xi])^{3/2}. \quad \square \end{aligned}$$

Corollary 1. For any fixed $\mu > 0$, $c_1 := 225\sqrt{n_1}$ and $c_2 := 225n_2^{3/2}$, $\eta(\mu, \cdot)$ is a μ -self-concordant function on \mathcal{F}_0 .

Proof. It is easy to verify that $\mu c_1 V_1$ is μ -self-concordant on \mathcal{F}_1 . The corollary follows from [18, Proposition 2.1.1]. \square

From here onwards we fix the values of c_1 and c_2 as in Corollary 1.

4.2 Parameters of the Self-Concordant Family $\eta(\mu, \cdot)$

Definition 2 (Nesterov and Nemirovskii [18, Definition 3.1.1]). Let \mathbb{R}^{++} be the set of all positive real numbers. Let G be an open nonempty convex subset of \mathbb{R}^n . Let $\mu \in \mathbb{R}^{++}$ and let $F_\mu : \mathbb{R}^{++} \times G \rightarrow \mathbb{R}$ be a family of functions indexed by μ . Let $\alpha_1(\mu)$, $\alpha_2(\mu)$, $\alpha_3(\mu)$, $\alpha_4(\mu)$ and $\alpha_5(\mu) : \mathbb{R}^{++} \rightarrow \mathbb{R}^{++}$ be continuously differentiable functions on μ . Then the family of functions $\{F_\mu\}_{\mu \in \mathbb{R}^{++}}$ is called strongly self-concordant with the parameters $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, if the following conditions hold:

- (i) F_μ is continuous on $\mathbb{R}^{++} \times G$, and for fixed $\mu \in \mathbb{R}^{++}$, F_μ is convex on G . F_μ has three partial derivatives on G , which are continuous on $\mathbb{R}^{++} \times G$ and continuously differentiable with respect to μ on \mathbb{R}^{++} .
- (ii) For any $\mu \in \mathbb{R}^{++}$, the function F_μ is strongly $\alpha_1(\mu)$ -self-concordant.
- (iii) For any $(\mu, x) \in \mathbb{R}^{++} \times G$ and any $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |\{\nabla_x F_\mu(\mu, x)[\xi]\}' - \{\ln \alpha_3(\mu)\}' \nabla_x F_\mu(\mu, x)[\xi]| &\leq \alpha_4(\mu) \alpha_1(\mu)^{\frac{1}{2}} (\nabla_{xx}^2 F_\mu(\mu, x)[\xi, \xi])^{\frac{1}{2}}, \\ |\{\nabla_{xx}^2 F_\mu(\mu, x)[\xi, \xi]\}' - \{\ln \alpha_2(\mu)\}' \nabla_{xx}^2 F_\mu(\mu, x)[\xi, \xi]| &\leq 2\alpha_5(\mu) \nabla_{xx}^2 F_\mu(\mu, x)[\xi, \xi]. \end{aligned}$$

Theorem 5. *The parametric function $\eta(\mu, \cdot)$ is a strongly self-concordant family with the following parameters*

$$\alpha_1(\mu) = \mu, \alpha_2(\mu) = \alpha_3(\mu) = 1, \alpha_4(\mu) = \mu^{-1}((m_1 c_1 + m_2 c_2)(1 + K))^{1/2}, \alpha_5(\mu) = \frac{1}{\mu}.$$

By Corollary 4, in order to prove Theorem 5, we only need to show that the two inequalities in Definition 2 (iii) are satisfied by $\eta(\mu, \cdot)$. We first show the validity of the second inequality in Definition 2 (iii).

Lemma 3. *For any $\mu > 0$ and $y \in \mathcal{F}_0$, the following inequality holds:*

$$|\nabla_{yy}^2 \eta'(\mu, y)[\xi, \xi]| \leq \frac{1}{\mu} \nabla_{yy}^2 \eta(\mu, y)[\xi, \xi], \quad \forall \xi \in \mathbb{R}^{m_1}.$$

Proof. Differentiating (28) with respect to μ and using (25) and (26) we obtain

$$\begin{aligned} \nabla_{yy}^2 \eta'(\mu, y) &= c_1 \nabla_y^2 V_1(y) + \sum_{k=1}^K c_2 (\nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)}) + \mu \nabla_{yy\bar{x}}^3 V_2^{(k)}(y, \bar{x}^{(k)}) \cdot (\bar{x}^{(k)})') \\ &= c_1 \nabla_{yy}^2 V_1(y) + \sum_{k=1}^K c_2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)}), \end{aligned}$$

which is exactly $\frac{1}{\mu} \nabla_{yy}^2 \eta(\mu, y)$. The conclusion follows since $\nabla_{yy}^2 \eta(\mu, y)$ is positive semidefinite. \square

For fixed (y, \bar{x}) with $S_2(y, \bar{x}) \succ 0$, let $\bar{T}_i = S_2^{-1/2} T_i S_2^{-1/2}$, $i = 1, 2, \dots, m_2$ and $\bar{W}_j = S_2^{-1/2} W_j S_2^{-1/2}$, $j = 1, 2, \dots, m_1$. We apply a Gram-Schmidt procedure to $\{\bar{T}_i\}$ to obtain $\{U_i\}$ with $\|U_i\| = 1$ for all i and $U_i \cdot U_j = 0, i \neq j$. Then the linear span of $\{U_i, i = 1, 2, \dots, m_2\}$ is equal to the span of $\{\bar{T}_i, i = 1, 2, \dots, m_2\}$. Let U be the $n_2^2 \times m_2$ matrix whose i^{th} column is $\text{vec}(U_i)$ and let $\Sigma = \sum_{k=1}^{m_2} U_k^2$. Then $P = P(S_2) = UU^\top$. We have

$$\begin{aligned} \frac{\partial V_2(y, \bar{x})}{\partial y_i} &= -P \cdot (\bar{W}_i \otimes_s I) = -UU^\top \cdot (\bar{W}_i \otimes_s I) = \text{trace}(U^\top (\bar{W}_i \otimes_s I) U) \\ &= -\frac{1}{2} \sum_{k=1}^{m_2} \text{vec}(U_k)^\top [(\bar{W}_i \otimes I) + (I \otimes \bar{W}_i)] \text{vec}(U_k) \\ &= -\frac{1}{2} \sum_{k=1}^{m_2} \text{vec}(U_k)^\top \text{vec}(U_k \bar{W}_i + \bar{W}_i U_k) \\ &= -\frac{1}{2} \sum_{k=1}^{m_2} U_k \cdot (U_k \bar{W}_i + \bar{W}_i U_k) = -\bar{W}_i \cdot \left(\sum_{k=1}^{m_2} U_k^2 \right) = -\bar{W}_i \cdot \Sigma. \end{aligned} \tag{36}$$

Similarly we have

$$\begin{aligned}
Q_{i,j}^{yy} &= P \cdot (\bar{W}_i \bar{W}_j \otimes_s I) = \text{trace}(U^\top (\bar{W}_i \bar{W}_j \otimes_s I) U) \\
&= \sum_{k=1}^{m_2} \text{vec}(U_k)^\top (\bar{W}_i \bar{W}_j \otimes_s I) \text{vec}(U_k) \\
&= \frac{1}{2} \sum_{k=1}^{m_2} \text{vec}(U_k)^\top \text{vec}(U_k \bar{W}_i \bar{W}_j + \bar{W}_i \bar{W}_j U_k) \\
&= \frac{1}{2} \sum_{k=1}^{m_2} U_k \cdot (U_k \bar{W}_i \bar{W}_j + \bar{W}_i \bar{W}_j U_k) \\
&= \frac{1}{2} \sum_{k=1}^{m_2} \text{trace}(U_k (U_k \bar{W}_i \bar{W}_j + \bar{W}_i \bar{W}_j U_k)) \\
&= \sum_{k=1}^{m_2} \text{trace}(\bar{W}_i U_k^2 \bar{W}_j) = \text{trace}(\bar{W}_i \Sigma \bar{W}_j).
\end{aligned} \tag{37}$$

Proposition 10. *Let (y, \bar{x}) be such that $S_2(y, \bar{x}) \succ 0$. Then*

$$\nabla_y V_2(y, \bar{x})^\top (\nabla_{yy}^2 V_2(y, \bar{x}))^{-1} \nabla_y V_2(y, \bar{x}) \leq m_2. \tag{38}$$

Proof. (See also [4].) Let \mathfrak{W} be the $n_2^2 \times m_1$ matrix whose i^{th} column is $\text{vec}(\bar{W}_i)$. Then from (37) we have

$$Q_{i,j}^{yy} = \text{trace}(\bar{W}_i \Sigma \bar{W}_j) = \text{vec}(\bar{W}_i)^\top \text{vec}(\Sigma \bar{W}_j) = \text{vec}(\bar{W}_i)^\top (I \otimes \Sigma) \text{vec}(\bar{W}_j).$$

We can then write $Q^{yy} = \mathfrak{W}^\top (I \otimes \Sigma) \mathfrak{W}$. Also it follows from (36) that $\nabla_y V_2(y, \bar{x})^\top = -\mathfrak{W}^\top \text{vec}(\Sigma)$. Hence

$$\begin{aligned}
\nabla_y V_2(y, \bar{x})^\top [Q^{yy}]^{-1} \nabla_y V_2(y, \bar{x}) &= \text{vec}(\Sigma)^\top \mathfrak{W} (\mathfrak{W}^\top (I \otimes \Sigma) \mathfrak{W})^{-1} \mathfrak{W}^\top \text{vec}(\Sigma) \\
&= \text{vec}(\Sigma^{1/2})^\top (I \otimes \Sigma^{1/2}) \mathfrak{W} (\mathfrak{W}^\top (I \otimes \Sigma) \mathfrak{W})^{-1} \mathfrak{W}^\top \\
&\quad (I \otimes \Sigma^{1/2}) \text{vec}(\Sigma^{1/2}) \\
&\leq \text{vec}(\Sigma^{1/2})^\top \text{vec}(\Sigma^{1/2}) = \text{trace}(\Sigma) = m_2,
\end{aligned}$$

since $\Sigma = \sum_{k=1}^{m_2} U_k^2$, and $\text{trace}(U_k^2) = U_k \cdot U_k = 1$ for each k . In addition, $Q^{yy} \preceq \nabla_y^2 V_2(y, \bar{x})$ implies $(\nabla_y^2 V_2(y, \bar{x}))^{-1} \preceq (Q^{yy})^{-1}$. So the result follows. \square

It can be easily verified that (38) is equivalent to the following inequality

$$|\nabla_y V_2(y, \bar{x})[\xi]| \leq \sqrt{m_2 \nabla_{yy}^2 V_2(y, \bar{x})[\xi, \xi]}, \quad \forall \xi \in \mathbb{R}^{m_1}. \tag{39}$$

Now we show the validity of the first inequality in Definition 2 (iii).

Lemma 4. *For any $\mu > 0$ and $y \in \mathcal{F}_0$, we have*

$$|\nabla_y \eta'(\mu, y)^\top[\xi]| \leq \sqrt{\frac{(m_1 c_1 + m_2 c_2)(1 + K)}{\mu}} \nabla_{yy}^2 \eta(\mu, y)[\xi, \xi], \quad \forall \xi \in \mathbb{R}^{m_1}.$$

Proof. From (28) and (24) we have

$$\begin{aligned}
\nabla_y \eta'(\mu, y) &= c_1 \nabla_y V_1(y) + \sum_{k=1}^K c_2 (\nabla_y V_2^{(k)}(y, \bar{x}^{(k)}) + \mu \nabla_{y\bar{x}}^2 V_2^{(k)}(y, \bar{x}^{(k)}) \cdot (\bar{x}^{(k)})') \\
&= c_1 \nabla_y V_1(y) + \sum_{k=1}^K c_2 \nabla_y V_2^{(k)}(y, \bar{x}^{(k)}).
\end{aligned}$$

Anstreicher [4, Theorem4.4] has shown that

$$\nabla_y V_1(y)^\top (\nabla_{yy}^2 V_1(y))^{-1} \nabla_y V_1(y) \leq m_1,$$

which is equivalent to

$$|\nabla_y V_1(y)[\xi]| \leq \sqrt{m_1 \nabla_{yy}^2 V_1(y)[\xi, \xi]}, \quad \forall \xi \in \mathbb{R}^{m_1}. \quad (40)$$

Then we have for all $\xi \in \mathbb{R}^{m_1}$ that

$$\begin{aligned} & |\nabla_y \eta'(\mu, y)^\top[\xi]| \\ &= |(c_1 \nabla_y V_1(y) + \sum_{k=1}^K c_2 \nabla_y V_2^{(k)}(y, \bar{x}^{(k)}))^\top[\xi]| \\ &\leq |c_1 \nabla_y V_1(y)^\top[\xi]| + \sum_{k=1}^K |c_2 \nabla_y V_2^{(k)}(y, \bar{x}^{(k)})^\top[\xi]| \\ &\leq \sqrt{m_1 c_1^2 \nabla_{yy}^2 V_1(y)[\xi, \xi]} + \sum_{k=1}^K \sqrt{m_2 c_2^2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)})[\xi, \xi]} \\ &= \sqrt{(m_1 c_1) c_1 \nabla_{yy}^2 V_1(y)[\xi, \xi]} + \sum_{k=1}^K \sqrt{(m_2 c_2) c_2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)})[\xi, \xi]} \\ &\leq \sqrt{(m_1 c_1 + m_2 c_2)(1 + K)(c_1 \nabla_{yy}^2 V_1(y)[\xi, \xi] + \sum_{k=1}^K c_2 \nabla_{yy}^2 V_2^{(k)}(y, \bar{x}^{(k)})[\xi, \xi])} \\ &= \sqrt{\frac{(m_1 c_1 + m_2 c_2)(1 + K)}{\mu} \nabla_{yy}^2 \eta(\mu, y)[\xi, \xi]}. \quad \square \end{aligned}$$

With Lemma 3 and Lemma 4 established, we have that Theorem 5 is true.

5 A Class of Volumetric Barrier Algorithms for Solving SSDP's

In §4 we have established that the parametric functions $\eta(\mu, \cdot)$ is a strongly self-concordant family. In this section we introduce a class of volumetric barrier algorithms for solving (9, 10). This class, indexed by a parameter $\gamma \in (0, 1)$, is stated formally in Algorithm 1.

Our algorithm is initialized with a starting point $y^0 \in \mathcal{F}_0$ and a starting value $\mu^0 > 0$ for the barrier parameter μ . We use δ as a measure of the proximity of the current point y to the central path, and β as a threshold for that measure. If the current y is too far away from the central path in the sense that $\delta > \beta$, we apply Newton's method to find a point close to the central path. Then we reduce the value of μ by a factor γ and repeat the whole process until the value of μ is within the tolerance ϵ .

6 Complexity Analysis

For fixed $\mu > 0$, the function $\eta(\mu, \cdot)$ possesses many nice properties. The following proposition follows directly from the definition of self-concordance and [18, Theorem 2.1.1].

Proposition 11. *For any $\mu > 0$, $y \in \mathcal{F}_0$ and Δy , we denote*

$$\delta := \sqrt{\frac{1}{\mu} \Delta y^\top \nabla_{yy}^2 \eta(\mu, y) \Delta y}. \text{ Then for } \delta < 1, \tau \in [0, 1] \text{ and any } \xi \in \mathbb{R}^{m_2} \text{ we have}$$

$$\xi^\top \nabla_{yy}^2 \eta(\mu, y + \tau \Delta y) \xi \leq (1 - \tau \delta)^{-2} \xi^\top \nabla_{yy}^2 \eta(\mu, y) \xi.$$

Algorithm 1 Volumetric Barrier Algorithm for Solving SSDP (9, 10)

Require: $\epsilon > 0$, $\gamma \in (0, 1)$, $\theta > 0$, $\beta > 0$, $y^0 \in \mathcal{F}_0$ and $\mu^0 > 0$.

$y := y^0$, $\mu := \mu^0$
while $\mu \geq \epsilon$ **do**
 for $k = 1, 2, \dots, K$ **do**
 solve (12) to obtain $\bar{x}^{(k)}$
 end for
 compute $\Delta y := -[\nabla_{yy}^2 \eta(\mu, y)]^{-1} \nabla_y \eta(\mu, y)$ using (28)
 compute $\delta(\mu, y) := \sqrt{\frac{1}{\mu} \Delta y^\top \nabla_{yy}^2 \eta(\mu, y) \Delta y}$ using (28)
 while $\delta > \beta$ **do**
 $y := y + \theta \Delta y$
 for $k = 1, 2, \dots, K$ **do**
 solve (12) to obtain $\bar{x}^{(k)}$
 end for
 compute $\Delta y := -[\nabla_{yy}^2 \eta(\mu, y)]^{-1} \nabla_y \eta(\mu, y)$ using (28)
 compute $\delta(\mu, y) := \sqrt{\frac{1}{\mu} \Delta y^\top \nabla_{yy}^2 \eta(\mu, y) \Delta y}$ using (28)
 end while
 $\mu := \gamma \mu$
end while

We also have the following lemma that describes the behavior of the Newton method as applied to $\eta(\mu, \cdot)$. This lemma is essentially a restatement of [18, Theorem 2.2.3] for the setting of the present paper.

Lemma 5. *For any $\mu > 0$ and $y \in \mathcal{F}_0$, let Δy be the Newton direction defined by $\Delta y := -[\nabla_{yy}^2 \eta(\mu, y)]^{-1} \nabla_y \eta(\mu, y)$. We denote*

$$\delta := \delta(\mu, y) = \sqrt{\frac{1}{\mu} \Delta y^\top \nabla_{yy}^2 \eta(\mu, y) \Delta y}.$$

Then the following relations hold:

(i) If $\delta < 2 - \sqrt{3}$ then

$$\delta(\mu, y + \Delta y) \leq \left(\frac{\delta}{1 - \delta} \right)^2 \leq \frac{\delta}{2}.$$

(ii) If $\delta \geq 2 - \sqrt{3}$ then

$$\eta(\mu, y) - \eta(\mu, y + \bar{\theta} \Delta y) \geq \mu[\delta - \ln(1 + \delta)],$$

where $\bar{\theta} = (1 + \delta)^{-1}$.

Depending on the manner in which γ in Algorithm 1 is chosen, we have two classes of algorithms: short-step algorithms and long-step algorithms. In the next two subsections we present the complexity analysis for these two classes of algorithms.

6.1 Complexity of Short Step Algorithms

The i^{th} iteration of the short-step algorithms is performed as follows: at the beginning of the iteration, we have $\mu^{(i-1)}$ and $y^{(i-1)}$ on hand and $y^{(i-1)}$ is close to the center path, i.e. $\delta(\mu^{(i-1)}, y^{(i-1)}) \leq \beta$. After we reduce the parameter μ from $\mu^{(i-1)}$ to $\mu^i := \gamma\mu^{(i-1)}$, we have that $\delta(\mu^i, y^{(i-1)}) \leq 2\beta$. Then a full Newton step is taken to find a new point y^i with $\delta(\mu^i, y^i) \leq \beta$. We will show that in each iteration after we reduce the parameter μ , one Newton step is sufficient to restore the proximity to the central path. We assume that we can solve all the subproblems exactly and we fix the value of $\gamma := 1 - 0.1/\sqrt{(m_1c_1 + m_2c_2)(1 + K)}$. We have the following lemma and the proof of the lemma follows from [18, Theorem 3.1.1].

Lemma 6. *Let $\gamma := 1 - 0.1/\sqrt{(m_1c_1 + m_2c_2)(1 + K)}$ and $\beta := (2 - \sqrt{3})/2$. If $\delta(\mu, y) \leq \beta$, then $\delta(\gamma\mu, y) \leq 2\beta$.*

Proof. In order to apply [18, Theorem 3.1.1], we first write the metric defined by (3.1.4) in [18] for our problem as follows: for any $0 < \mu^+ < \mu$,

$$\chi_\kappa(\eta; \mu, \mu^+) := \left(1 + \frac{\sqrt{(m_1c_1 + m_2c_2)(1 + K)}}{\kappa} \right) \ln \left(\frac{\mu}{\mu^+} \right).$$

Let $\kappa := 2\beta := 2 - \sqrt{3}$. Since $\delta(\mu, y) \leq \kappa/2$, one can verify that $\mu^+ := \gamma\mu$ satisfies

$$\chi_\kappa(\eta; \mu, \mu^+) \leq \frac{1}{2} \leq 1 - \frac{\delta(\mu, y)}{\kappa}.$$

So by virtue of [18, Theorem 3.1.1], we have $\delta(\mu^+, y) \leq \kappa$. □

By inequality in Lemma 5(i) and Lemma 6 we have that in Algorithm 1, we can reduce the parameter μ by the factor $\gamma := 1 - 0.1/\sqrt{(m_1c_1 + m_2c_2)(1 + K)}$ at each iteration, and use only one Newton step for recentering if necessary. So we have the following complexity result for short-step algorithms.

Theorem 6. *Let $\beta := (2 - \sqrt{3})/2$ and $\gamma := 1 - 0.1/\sqrt{(m_1c_1 + m_2c_2)(1 + K)}$ in Algorithm 1. If $\delta(\mu^0, y^0) \leq \beta$, then short-step algorithms terminate with at most $O(\sqrt{(m_1c_1 + m_2c_2)(1 + K)} \ln(\mu^0/\epsilon))$ iterations.*

6.2 Complexity of the Long Step Algorithms

In the long-step version of the algorithm, the factor $\gamma \in (0, 1)$ is arbitrarily chosen. It has potential for larger decrease on the objective function value, however, several damped Newton steps might be needed for recentering.

Suppose at the beginning of the i^{th} iteration of the algorithm we have a point $y^{(i-1)}$, which is sufficiently close to $y(\mu^{(i-1)})$, where $\mu^{(i-1)}$ is the current value for the barrier parameter μ and $y(\mu^{(i-1)})$ is the solution to (11) for $\mu := \mu^{(i-1)}$. We reduce the barrier parameter from $\mu^{(i-1)}$ to $\mu^i := \gamma\mu^{(i-1)}$, where $\gamma \in (0, 1)$, and we search for a point y^i that is sufficiently close to $y(\mu^i)$. We want to determine an upper bound on the number of Newton iterations that are needed to find the point y^i .

We begin by defining $\phi(\mu) := \eta(\mu, y) - \eta(\mu, y(\mu))$, $\forall \mu > 0$.

The function ϕ stands for the difference between the objective value $\eta(\mu^i, y^{(i-1)})$ at the beginning of i^{th} iteration and the minimum objective value $\eta(\mu^i, y(\mu^i))$ at the end of i^{th} iteration. Then our task is to find an upper bound on $\phi(\mu)$.

The next lemma gives us upper bounds on $\phi(\mu)$ and $\phi'(\mu)$ respectively.

Lemma 7. *let $\mu > 0$ and $y \in \mathcal{F}_0$, we denote $\tilde{\Delta}y := y - y(\mu)$ and define*

$$\tilde{\delta} := \sqrt{\frac{1}{\mu} \tilde{\Delta}y^\top \nabla_{yy}^2 \eta(\mu, y) \tilde{\Delta}y}.$$

If $\tilde{\delta} < 1$, then the following inequalities hold:

$$\phi(\mu) \leq \left[\frac{\tilde{\delta}}{1 - \tilde{\delta}} + \ln(1 - \tilde{\delta}) \right] \mu, \quad (41)$$

$$|\phi'(\mu)| \leq -\ln(1 - \tilde{\delta}) \sqrt{(m_1 c_1 + m_2 c_2)(1 + K)}. \quad (42)$$

Proof.

$$\phi(\mu) := \eta(\mu, y) - \eta(\mu, y(\mu)) := \int_0^1 \nabla_y \eta(\mu, y(\mu) + (1 - \tau)\tilde{\Delta}y)^\top \tilde{\Delta}y d\tau.$$

Since $y(\mu)$ is the optimal solution, we have

$$\nabla_y \eta(\mu, y(\mu)) = 0. \quad (43)$$

Hence

$$\begin{aligned} \phi(\mu) &= \int_0^1 \int_0^1 \tilde{\Delta}y^\top \nabla_{yy}^2 \eta(\mu, y(\mu) + (1 - \alpha)\tilde{\Delta}y) \tilde{\Delta}y d\alpha d\tau \\ &\leq \int_0^1 \int_0^1 \frac{\mu \tilde{\delta}^2}{(1 - \tilde{\delta} + \alpha \tilde{\delta})^2} d\alpha d\tau \quad (\text{by Proposition 11}) \\ &= \left[\frac{\tilde{\delta}}{1 - \tilde{\delta}} + \ln(1 - \tilde{\delta}) \right] \mu. \end{aligned}$$

This proves (41).

For any $\mu > 0$,

$$\begin{aligned} \phi'(\mu) &= \eta'(\mu, y) - \eta'(\mu, y(\mu)) - \nabla \eta(\mu, y(\mu))^\top y'(\mu) \\ &= \eta'(\mu, y) - \eta'(\mu, y(\mu)). \end{aligned} \quad (44)$$

From Lemma 4 we have that for any $\mu > 0$ and $y \in \mathcal{F}_0$

$$\nabla \eta'(\mu, y)^\top [\nabla_{yy}^2 \eta(\mu, y)]^{-1} \nabla \eta'(\mu, y) \leq \frac{1}{\mu} (m_1 c_1 + m_2 c_2)(1 + K). \quad (45)$$

Applying the Mean-Value Theorem we obtain

$$\begin{aligned} |\phi'(\mu)| &= \left| \int_0^1 \nabla \eta'(\mu, y(\mu) + \tau \tilde{\Delta}y)^\top \tilde{\Delta}y d\tau \right| \\ &\leq \int_0^1 \sqrt{\tilde{\Delta}y^\top \nabla_{yy}^2 \eta(\mu, y(\mu) + \tau \tilde{\Delta}y) \tilde{\Delta}y} \\ &\quad \sqrt{\nabla \eta'(\mu, y(\mu) + \tau \tilde{\Delta}y)^\top [\nabla_{yy}^2 \eta(\mu, y(\mu) + \tau \tilde{\Delta}y)]^{-1} \nabla \eta'(\mu, y(\mu) + \tau \tilde{\Delta}y) d\tau}. \end{aligned}$$

Using (45) and Proposition 11, and noting that $y(\mu) + \tau\tilde{\Delta}y = y - (1 - \tau)\tilde{\Delta}y$, we have

$$\begin{aligned} |\phi'(\mu)| &\leq \int_0^1 \frac{\tilde{\delta}\sqrt{\mu}}{1 - \tilde{\delta} + \tau\tilde{\delta}} \sqrt{\frac{(m_1c_1 + m_2c_2)(1 + K)}{\mu}} d\tau \\ &= -\ln(1 - \tilde{\delta})\sqrt{(m_1c_1 + m_2c_2)(1 + K)}. \end{aligned}$$

This proves (42). \square

Lemma 8. *Let $\mu > 0$ and $y \in \mathcal{F}_0$ be such that $\tilde{\delta} < 1$, where $\tilde{\delta}$ is as defined in Lemma 7. Let $\mu^+ := \gamma\mu$ with $\gamma \in (0, 1)$. Then*

$$\eta(\mu^+, y) - \eta(\mu^+, y(\mu^+)) \leq O(1)[(m_1c_1 + m_2c_2)(1 + K)]\mu^+.$$

Proof. We have

$$\phi''(\mu, y) = \eta''(\mu, y) - \eta''(\mu, y(\mu)) - \nabla\eta'(\mu, y(\mu))^\top y'(\mu).$$

Since $\eta(\cdot, y)$ is strictly concave in μ , the first term $\eta''(\mu, y)$ is negative. We only need to estimate the other two terms.

First we differentiate (43) with respect to μ to obtain

$$y'(\mu) = -[\nabla_{yy}^2\eta(\mu, y(\mu))]^{-1}\nabla\eta'(\mu, y(\mu)).$$

Hence from Lemma 4 we have the following estimation

$$\begin{aligned} -\nabla\eta'(\mu, y(\mu))^\top y'(\mu) &= \nabla\eta'(\mu, y(\mu))^\top [\nabla_{yy}^2\eta(\mu, y(\mu))]^{-1}\nabla\eta'(\mu, y(\mu)) \\ &\leq \frac{1}{\mu}(m_1c_1 + m_2c_2)(1 + K). \end{aligned} \quad (46)$$

Now we want to estimate $\eta''(\mu, y)$ for any $\mu > 0$ and $y \in \mathcal{F}_0$. First we observe that $\eta'' = \sum_{k=1}^K \rho_k''(\mu, y)$.

Differentiating $\rho_k(\mu, y)$ with respect to μ we obtain $\rho_k'(\mu, y) = c_2V_2(y, \bar{x})$. Differentiating again we obtain $\rho_k''(\mu, y) = c_2\nabla_{\bar{x}}V_2(y, \bar{x})^\top \bar{x}'$. We differentiate (24) to obtain

$$\bar{x}' = -\frac{1}{\mu}[\nabla_{\bar{x}}^2V_2(y, \bar{x})]^{-1}\nabla_{\bar{x}}V_2(y, \bar{x}).$$

Hence we have

$$\rho_k''(\mu, y) = -\frac{1}{\mu}c_2\nabla_{\bar{x}}V_2(y, \bar{x})^\top [\nabla_{\bar{x}}^2V_2(y, \bar{x})]^{-1}\nabla_{\bar{x}}V_2(y, \bar{x}).$$

By [4, Theorem 4.4] we have $-\rho_k''(\mu, y) \leq \frac{1}{\mu}c_2m_2$. Therefore

$$-\eta''(\mu, y(\mu)) = -\sum_{k=1}^K \rho_k''(\mu, y) \leq \sum_{k=1}^K \frac{1}{\mu}c_2m_2 = \frac{1}{\mu}m_2c_2K. \quad (47)$$

Combining (46) and (47), we have

$$\phi''(\mu) \leq \frac{1}{\mu}(m_1c_1 + 2m_2c_2)(1 + K). \quad (48)$$

Using Lemma 7 and (48), we have

$$\begin{aligned}
\phi(\mu^+) &= \phi(\mu) + (\mu^+ - \mu)\phi'(\mu) + \int_{\mu^+}^{\mu} \int_{\tau}^{\mu} \phi''(\mu) d\mu d\tau \\
&\leq \left[\frac{\tilde{\delta}}{1 - \tilde{\delta}} + \ln(1 - \tilde{\delta}) \right] \mu - \ln(1 - \tilde{\delta}) \sqrt{(m_1 c_1 + m_2 c_2)(1 + K)} (\mu^+ - \mu) \\
&\quad + (m_1 c_1 + 2m_2 c_2)(1 + K) \int_{\mu^+}^{\mu} \int_{\tau}^{\mu} \mu^{-1} d\mu d\tau \\
&\leq \left[\frac{\tilde{\delta}}{1 - \tilde{\delta}} + \ln(1 - \tilde{\delta}) \right] \mu - \ln(1 - \tilde{\delta}) \sqrt{(m_1 c_1 + m_2 c_2)(1 + K)} (\mu^+ - \mu) \\
&\quad + (m_1 c_1 + 2m_2 c_2)(1 + K) \ln \gamma^{-1} (\mu - \mu^+).
\end{aligned}$$

This proves the lemma since $\tilde{\delta}$ and γ are constants. \square

In the previous lemmas we require $\tilde{\delta} < 1$. However $\tilde{\delta}$ cannot be evaluated explicitly. In the next lemma we will see that $\tilde{\delta}$ is actually proportional to δ , which can be evaluated.

Lemma 9. *For any $\mu > 0$ and $y \in \mathcal{F}_0$, let $\Delta y := -[\nabla_{yy}^2 \eta(\mu, y)]^{-1} \nabla_y \eta(\mu, y)$ and let $\tilde{\Delta} y := (y - y(\mu))$. We denote*

$$\delta := \sqrt{\frac{1}{\mu} \Delta y^\top \nabla_{yy}^2 \eta(\mu, y) \Delta y}, \quad \tilde{\delta} := \sqrt{\frac{1}{\mu} \tilde{\Delta} y^\top \nabla_{yy}^2 \eta(\mu, y) \tilde{\Delta} y}.$$

If $\delta < 1/6$, then

$$\frac{2}{3} \delta \leq \tilde{\delta} \leq 2\delta.$$

Proof. See the proof of [4, Lemma 9]. \square

Combining Lemmas 5, 8, and 9, we have the following theorem.

Theorem 7. *Let $\beta := 1/6$ and $\gamma \in (0, 1)$ be arbitrary in Algorithm 1. If $\delta(\mu^0, y^0) \leq \beta$, then long-step algorithms terminate with at most $O((m_1 c_1 + m_2 c_2)(1 + K) \ln(\mu^0/\epsilon))$ iterations.*

7 Concluding Remarks

In this paper we have presented a class of volumetric barrier decomposition algorithms for (two-stage) stochastic semidefinite programs (SSDP's) (with recourse). We have also shown that certain short-step and long-step members of the class have polynomial complexity in terms of the number of iterations with the complexity bounds depending on \sqrt{K} and K respectively, where K is the number of realizations. This is important given the fact K can be large in applications.

SSDP's generalize (two-stage) stochastic linear programs (SLP's) (with recourse). Therefore, it is possible to specialize the class of algorithms presented in this paper to SLP's. The specialization is a new class of algorithms for SLP's. In a forthcoming paper we show that we can go further by showing how appropriate modification of the techniques utilized on the present paper leads to a class of new volumetric barrier decomposition algorithms for stochastic quadratic programs with quadratic recourse.

It would be interesting to assess the computational performance of the algorithms developed in the present paper. A forthcoming paper will report details of an implementation and results of computational experiments performed with it.

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