Exam 1 Key

1. (a) **L** is lower triangular with 1's on the diagonal. **U** is upper diagonal with pivots on the diagonal.

   (b) No. If a zero pivot is encountered during Gaussian elimination, then a nonsingular matrix will not have a **LU** factorization. (However, a permuted version of the matrix would have a **LU** factorization.)

   (c) If the **LU** factorization of a matrix is known, solving the linear system **Ax** = **b** using this factorization involves far fewer arithmetic operations than Gaussian elimination.

   (d) **Ax** = (**LU**)**x** = **L**(**Ux**) = **Ly** = **b**, where **Ux** = **y**. To solve **Ax** = **b**, first solve **Ly** = **b** (forward substitution): $y_1 = b_1 = 1$; $2y_1 + y_2 = 2(1) + y_2 = b_2 = -1$ $y_2 = -3$. Next, solve **Ux** = **y** (back substitution): $3x_2 = y_2 = -3$ $x_2 = -1$. $2x_1 - x_2 = 2x_1 - (-1) = y_1 = 1$ $2x_1 = 0$ $x_1 = 0$. $\mathbf{x} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

2. (a) Multiply both sides of $\mathbf{A}(\mathbf{I}-\mathbf{A}) = (\mathbf{I}-\mathbf{A})\mathbf{A}$ on the left by $(\mathbf{I}-\mathbf{A})^{-1}$ $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{A}(\mathbf{I}-\mathbf{A}) = \mathbf{A}$. Now multiply both sides by $(\mathbf{I}-\mathbf{A})^{-1}$ on the right $(\mathbf{I}-\mathbf{A})^{-1}\mathbf{A} = \mathbf{A}(\mathbf{I}-\mathbf{A})^{-1}$.

   (b)

$$\mathbf{B}^{\mathrm{T}} = \left[(\mathbf{I}+\mathbf{K})(\mathbf{I}-\mathbf{K})^{-1}\right]^{\mathrm{T}} = \left[(\mathbf{I}-\mathbf{K})^{-1}\right]^{\mathrm{T}}(\mathbf{I}+\mathbf{K})^{\mathrm{T}} = \left[(\mathbf{I}-\mathbf{K})^{\mathrm{T}}\right]^{-1}(\mathbf{I}+\mathbf{K})^{\mathrm{T}}$$
$$= \left(\mathbf{I}^{\mathrm{T}}-\mathbf{K}^{\mathrm{T}}\right)^{-1}\left(\mathbf{I}^{\mathrm{T}}+\mathbf{K}^{\mathrm{T}}\right) = (\mathbf{I}+\mathbf{K})^{-1}(\mathbf{I}-\mathbf{K})$$

$$\mathbf{B}^{-1} = \left[(\mathbf{I}+\mathbf{K})(\mathbf{I}-\mathbf{K})^{-1}\right]^{-1} = \left[(\mathbf{I}-\mathbf{K})^{-1}\right]^{-1}(\mathbf{I}+\mathbf{K})^{-1}$$
$$= (\mathbf{I}-\mathbf{K})(\mathbf{I}+\mathbf{K})^{-1} \overset{\text{by (a)}}{=} (\mathbf{I}+\mathbf{K})^{-1}(\mathbf{I}-\mathbf{K}) = \mathbf{B}^{\mathrm{T}}$$

3. (a) Use Gaussian elimination to solve $\mathbf{Ax} = \mathbf{0}$:

$$(\mathbf{A}\,|\,\mathbf{0}) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \underline{1} & 1 & -1 & 0 \\ 0 & \underline{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$ Now solve

for the basic variables (x_1 and x_2) in terms of the free variable x_3: $x_2 + x_3 = 0$ $\Rightarrow$

$x_2 = -x_3$. $x_1 + x_2 - x_3 = x_1 + (-x_3) - x_3 = x_1 - 2x_3 = 0$ $\Rightarrow$ $x_1 = 2x_3$

$$\mathbf{x} = \begin{pmatrix} 2x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \text{ a basis for the nullspace of } \mathbf{A} \text{ is } \left\{\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right\}.$$

(b) $\dim R(\mathbf{A}^T) = \operatorname{rank}(\mathbf{A}^T) = \operatorname{rank}(\mathbf{A}) = 2$ since the row echelon form of $\mathbf{A}$ has two pivots (see underlined elements above).

4. $\tilde{S} = \{\mathbf{w}, \mathbf{z}\}$ is linearly independent $\Leftrightarrow$ $\alpha_1\mathbf{w} + \alpha_2\mathbf{z} = \mathbf{0}$ $\Rightarrow$ $\alpha_1 = \alpha_2 = 0$. Now $\alpha_1\mathbf{w} + \alpha_2\mathbf{z} = \alpha_1(\mathbf{x} + \mathbf{y}) + \alpha_2(\mathbf{x} - \mathbf{y}) = (\alpha_1 + \alpha_2)\mathbf{x} + (\alpha_1 - \alpha_2)\mathbf{y} = \mathbf{0}$. The coefficients of $\mathbf{x}$ and $\mathbf{y}$ must both vanish since S is a linearly independent set. That is, $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 0$ $\Rightarrow$ $\alpha_1 = -\alpha_2$ and $\alpha_1 = \alpha_2$ $\Rightarrow$ $\alpha_2 = -\alpha_2$ $\Rightarrow$ $\alpha_2 = 0$ $\Rightarrow$ $\alpha_1 = 0$. Thus, $\alpha_1\mathbf{w} + \alpha_2\mathbf{z} = \mathbf{0}$ $\Rightarrow$ $\alpha_1 = \alpha_2 = 0$ and therefore, $\tilde{S}$ is a linearly independent set.

5. Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 1 & -1 \end{pmatrix}$. Then $\mathbf{v} \in \operatorname{span}(S)$ $\Leftrightarrow$ $\mathbf{Ax} = \mathbf{v}$ is consistent $\Leftrightarrow$ $\operatorname{rank}(\mathbf{A}|\mathbf{v}) = \operatorname{rank}(\mathbf{A})$.

Now $$(\mathbf{A}\,|\,\mathbf{v}) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & c \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & -2 & c-1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & 0 & c+3 \end{array}\right).$$ Thus, $\operatorname{rank}(\mathbf{A}|\mathbf{v})$ $= \operatorname{rank}(\mathbf{A})$ $\Leftrightarrow$ $c + 3 = 0$ $\Leftrightarrow$ $c = -3$. Thus, $\mathbf{v} \in \operatorname{span}(S)$ if and only if $c = -3$.