Exam 2 Key

1. Consider

$$\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle$$

$$= \|\mathbf{x}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle + \|\mathbf{y}\|^2 \text{ since } \langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle \text{ for real spaces.}$$

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 \Leftrightarrow \langle \mathbf{x}, \mathbf{y}\rangle = 0 \Leftrightarrow \mathbf{x} \quad \mathbf{y}.$$

2. By definition $\|\mathbf{U}\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{U}\mathbf{x}\|_2$. But $\mathbf{U}$ unitary $\quad \|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2 \quad \|\mathbf{U}\|_2 = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{x}\|_2 = 1$.

3. If (i) holds, $\mathbf{b} \quad R(\mathbf{A})$. Since $R(\mathbf{A}) \quad = N(\mathbf{A}^\mathrm{T})$, $\langle \mathbf{y}, \mathbf{b}\rangle = 0 \quad \mathbf{y} \quad N(\mathbf{A}^\mathrm{T})$. That is, $\langle \mathbf{y}, \mathbf{b}\rangle = 0 \quad \mathbf{y}$ such that $\mathbf{y}^\mathrm{T}\mathbf{A} = \mathbf{0}$. Thus, if (i) is true, (ii) must be false. If (ii) holds, then $\mathbf{b} \quad N(\mathbf{A}) \quad = R(\mathbf{A})$ there is no $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$. Thus, if (ii) is true, (i) must be false.

4. (a) $F(\mathbf{U}_1) = \mathbf{U}_1$, $F(\mathbf{U}_2) = \mathbf{U}_3$, $F(\mathbf{U}_3) = \mathbf{U}_2$, and $F(\mathbf{U}_4) = \mathbf{U}_4$

$$[F]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) Need to compute $[F(\mathbf{A})]_B = [F]_B\,[\mathbf{A}]_B$. But $\mathbf{A} = a\mathbf{U}_1 + b\mathbf{U}_2 + c\mathbf{U}_3 + d\mathbf{U}_4 \quad [\mathbf{A}]_B =$

$$\begin{pmatrix} a & b & c & d \end{pmatrix}^\mathrm{T} \quad [F(\mathbf{A})]_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \quad F(\mathbf{A}) = a\mathbf{U}_1 + c\mathbf{U}_2 + b\mathbf{U}_3 + d\mathbf{U}_4 =$$

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

4. (c) $[\mathbf{V}_1]_B = \mathbf{U}_1 + \mathbf{U}_4$, $[\mathbf{V}_2]_B = \mathbf{U}_2 + \mathbf{U}_3$, $[\mathbf{V}_3]_B = \mathbf{U}_1 + \mathbf{U}_3 + \mathbf{U}_4$, and $[\mathbf{V}_1]_B = \mathbf{U}_1 + \mathbf{U}_2 + \mathbf{U}_4$

$$[I]_{B'B} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}.$$

5. (a) $X + Y$ is spanned by $B = B_X \cup B_Y = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$. Since $\mathbf{Q}$ is orthogonal, B is orthonormal which implies B is linearly independent. Thus, B is a basis for $X + Y$. Since $\mathbf{Q}$ is nonsingular, B must also be a basis for $\mathbb{R}^3$. Thus, $\mathbb{R}^3 = X + Y$. To show $X \cap Y = \{\mathbf{0}\}$, let $\mathbf{v} \in X \cap Y$. Then $\mathbf{v} = a\mathbf{q}_1 + b\mathbf{q}_2$ for some a and b since $\mathbf{v} \in X$ and $\mathbf{v} = c\mathbf{q}_3$ for some c since $\mathbf{v} \in Y$. This implies $\|\mathbf{v}\|^2 = \langle \mathbf{v}, \mathbf{v} \rangle = \langle a\mathbf{q}_1 + b\mathbf{q}_2, c\mathbf{q}_3 \rangle = ac\langle \mathbf{q}_1, \mathbf{q}_3 \rangle + bc\langle \mathbf{q}_2, \mathbf{q}_3 \rangle = 0$ since the $\mathbf{q}_I$'s are orthogonal. Thus, $\mathbf{v} = \mathbf{0} \Rightarrow X \cap Y = \{\mathbf{0}\} \Rightarrow \mathbb{R}^3 = X \oplus Y$.

(b) Use the Fourier expansion of $\mathbf{v}$: $\mathbf{v} = \underbrace{\left(\langle \mathbf{v}, \mathbf{q}_1 \rangle \mathbf{q}_1 + \langle \mathbf{v}, \mathbf{q}_2 \rangle \mathbf{q}_2\right)}_{X} + \underbrace{\langle \mathbf{v}, \mathbf{q}_3 \rangle \mathbf{q}_3}_{Y} = \underbrace{\left(3\mathbf{q}_1 - \mathbf{q}_2\right)}_{X} + \underbrace{\left(-\mathbf{q}_3\right)}_{Y}$

Thus $\mathbf{v} = \mathbf{x} + \mathbf{y}$ where $\mathbf{x} = 3\mathbf{q}_1 - \mathbf{q}_2 = \begin{pmatrix} 3/\sqrt{2} \\ -1 \\ 3/\sqrt{2} \end{pmatrix} \in X$ and $\mathbf{y} = -\mathbf{q}_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$.