Key to Problem Set #2

1. Using Gauss-Jordan elimination, one reduces the augmented matrix $(\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$ using elementary row operations (details not shown) to $\left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) = \left(\mathbf{I} \mid \mathbf{A}^{-1}\right)$ where $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

2. In each case, the inverse reverses the corresponding row operation:

   (i) $\mathbf{P}_{ij}^{-1} = \mathbf{I} + \frac{(\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}}}{1 - (\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}}(\mathbf{e}_i - \mathbf{e}_j)} = \mathbf{I} - (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}} = \mathbf{P}_{ij}$. That is, the inverse of $\mathbf{P}_{ij}$ also exchanges rows i and j.

   (ii) If $\mathbf{D}_i = \mathbf{I} - (1-\alpha)\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$, then $\mathbf{D}_i$ multiplies row i by α. We have $\mathbf{D}_i^{-1} = \mathbf{I} + \frac{(1-\alpha)\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}}{1-(1-\alpha)\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_i} = \mathbf{I} - \left(1 - \frac{1}{\alpha}\right)\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$. That is, $\mathbf{D}_i^{-1}$ <u>divides</u> row i by α.

   (iii) If $\mathbf{E}_{ij} = \mathbf{I} + \alpha\,\mathbf{e}_j\mathbf{e}_i^{\mathrm{T}}$, then α times row i is added to row j. We have $\mathbf{E}_{ij}^{-1} = \left[\mathbf{I} - (-\alpha)\mathbf{e}_j\mathbf{e}_i^{\mathrm{T}}\right]^{-1} = \mathbf{I} + \frac{-\alpha\,\mathbf{e}_j\mathbf{e}_i^{\mathrm{T}}}{1-(-\alpha)\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j} = \mathbf{I} + (-\alpha)\mathbf{e}_j\mathbf{e}_i^{\mathrm{T}}$. That is, $\mathbf{E}_{ij}^{-1}$ <u>subtracts</u> α times row i from row j.

3. The following give some possibilities (many other answers are correct for parts a, b, and c):

   (a) Suppose **A** is nonsingular. Then **B** = **-A** is also nonsingular (since a scalar multiple of a nonsingular matrix is nonsingular) but **A** + **B** = **A** – **A** = **0** is clearly not invertible.

   (b) The matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is singular since it is in row echelon form and has only one pivot. Likewise, the matrix $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is singular since a row exchange shows it has a only one pivot. But $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ has rank = 2 and is thus invertible.

(c) Both **A** = **I** and **B** = 2**I** are obviously nonsingular. **A** + **B** = 3**I** is also nonsingular.

(d) Since $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$, and **A**+**B** are all nonsingular matrices by assumption, then $\mathbf{B}^{-1}+\mathbf{A}^{-1}=\mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1}$ is the product of nonsingular matrices and is thus nonsingular as well. $\left(\mathbf{B}^{-1}+\mathbf{A}^{-1}\right)^{-1}=\left[\mathbf{A}^{-1}(\mathbf{A}+\mathbf{B})\mathbf{B}^{-1}\right]^{-1}=\mathbf{B}(\mathbf{A}+\mathbf{B})^{-1}\mathbf{A}$.

4. $\kappa_{\mathbf{A}}=\frac{4}{b}+4+b$, $\kappa_{\mathbf{B}}=\frac{4}{b}+4+b$, $\kappa_{\mathbf{C}}=\frac{4}{b}$, and $\kappa_{\mathbf{D}}=\frac{4}{b^2}+\frac{4}{b}+1$. Since $0 < b < 1$, we have $\kappa_{\mathbf{D}}>\kappa_{\mathbf{A}}=\kappa_{\mathbf{B}}>\kappa_{\mathbf{C}}$. Matrix **D** is an order of magnitude more ill conditioned than the other matrices. Matrix **C** is only marginally better conditioned than **A** and **B**.