# Key to Problem Set #3

1. (a) For any linear transformation **T** and basis $C = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, the $i$th column of the coordinate matrix $[\mathbf{T}]_C$ consists of the coordinates of $\mathbf{T}(\mathbf{v}_i)$ with respect to the basis $C$, where $\mathbf{v}_i$ is the $i$th basis vector. For the identity transformation, $\mathbf{I}(\mathbf{v}_i) = \mathbf{v}_i$ $= 0\ \mathbf{v}_1 + \cdots + 0\ \mathbf{v}_{i-1} + 1\ \mathbf{v}_i + 0\ \mathbf{v}_{i+1} + \cdots 0\ \mathbf{v}_n$. Thus, the $i$th column of $[\mathbf{I}]_C = \mathbf{e}_i$ for $i = 1, \ldots, n$ which implies that $[\mathbf{I}]_C = \mathbf{I}_n$. This argument holds when $C = B$ and when $C = B'$.

   (b) The $j$th column of $[\mathbf{I}]_{BB'}$ contains the coefficients of $\mathbf{e}_j$ when expanded with respect to the basis $B'$, that is, the coefficients $x_{ij}$ in the expansion $\mathbf{e}_j = x_{1j}\mathbf{u}_1 + x_{2j}\mathbf{u}_2 + \cdots + x_{nj}\mathbf{u}_n$. If $\mathbf{A} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n]$ and $\mathbf{x}_j = [x_{ij}]$, then the coefficients satisfy $\mathbf{A}\mathbf{x}_j = \mathbf{e}_j$ $\quad \mathbf{x}_j = \mathbf{A}^{-1}\mathbf{e}_j$ ($\mathbf{A}_{n\times n}$ must be nonsingular since $B'$ is a basis for $\ ^n$) $\quad \mathbf{x}_j$ is the $j$th column of $\mathbf{A}^{-1}$. Thus, $[\mathbf{I}]_{BB'} = \mathbf{A}^{-1}$.

   (c) $[\mathbf{I}]_{B'B} = \{[\mathbf{I}]_{BB'}\}^{-1} = \{\mathbf{A}^{-1}\}^{-1} = \mathbf{A} = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \cdots \mid \mathbf{u}_n]$.

2. (a) $[T]_B = \begin{pmatrix} 3 & 0 \\ 0 & 11 \end{pmatrix}$

   (b) $[x]_B = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \quad [T(x)]_B = \begin{pmatrix} 3 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 9 \\ -11 \end{pmatrix}$. Thus, $T(x) = 9e^{2t} - 11e^{-2t}$.

   (c) $[\mathbf{I}]_{BB'} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

   (d) $[T]_{B'} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 11 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 7 & -4 \\ -4 & 7 \end{pmatrix}$

3. Let **Q** = **B**. Since, by assumption, **B** is nonsingular, we have $\mathbf{Q}^{-1}(\mathbf{BA})\mathbf{Q} = \mathbf{B}^{-1}(\mathbf{BA})\mathbf{B} = \mathbf{B}^{-1}\mathbf{BAB} = \mathbf{IAB} = \mathbf{AB}$. So there exists a nonsingular matrix **Q** such that $\mathbf{AB} = \mathbf{Q}^{-1}(\mathbf{BA})\mathbf{Q}$ and thus **AB** is similar to **BA**.

4. By the CBS inequality, $|\mathbf{a}^{\mathrm{T}}\mathbf{b}| \le \|\mathbf{a}\|\ \|\mathbf{b}\|$. But $\mathbf{a}^{\mathrm{T}}\mathbf{b} = 2\sqrt{xy}$, $\|\mathbf{a}\| = \sqrt{(\sqrt{x})^2 + (\sqrt{y})^2} = \sqrt{x+y}$, and $\|\mathbf{b}\| = \sqrt{(\sqrt{y})^2 + (\sqrt{x})^2} = \sqrt{x+y}$. So the CBS inequality is (taking positive square roots) $2\sqrt{xy} \le \sqrt{x+y}\ \sqrt{x+y}$ or $\sqrt{xy} \le \frac{1}{2}(x+y)$. That is, the geometric mean of two numbers is never larger than their arithmetic mean.