Key to Problem Set #4

1. (a) Obviously, $|f(t)| \geq 0$ for all $t \in [0, 1] \Rightarrow \|f\| \geq 0$. If $f(t) = 0$ for all $t \in [0, 1]$ then clearly $\|f\| = 0$. If $\|f\| = 0$, then $0 = \max_{t \in [0,1]} |f(t)| \geq |f(t)| \geq 0 \Rightarrow |f(t)| = 0$ for all $t \in [0, 1] \Rightarrow f(t) = 0$ for all $t \in [0, 1]$. Thus, $\|f\| = 0 \Leftrightarrow f(t) \equiv 0$. Now let α be a real scalar and consider $\|\alpha f\| = \max_{t \in [0,1]} |\alpha f(t)| = |\alpha| \max_{t \in [0,1]} |f(t)| = |\alpha| \|f\|$. Finally, $\|f + g\| = \max_{t \in [0,1]} |f(t) + g(t)| \leq \max_{t \in [0,1]} \left(|f(t)| + |g(t)|\right) \leq \max_{t \in [0,1]} |f(t)| + \max_{t \in [0,1]} |g(t)| = \|f\| + \|g\|$ which proves the triangle inequality.

   (b) There is an inner product if and only if $\|f + g\|^2 + \|f - g\|^2 = 2\left(\|f\|^2 + \|g\|^2\right)$ for all f and g in V. Consider $f(t) = 1$ and $g(t) = t$. $\|f + g\|^2 = 4$, $\|f - g\|^2 = 1$, $\|f\| = 1$, and $\|g\| = 1$. Thus $\|f + g\|^2 + \|f - g\|^2 = 5 \neq 2\left(\|f\|^2 + \|g\|^2\right) = 4$ which implies that there is no inner product on V such that $\langle f, f \rangle = \|f\|^2$ for all $f \in V$.

2. (a) $$\theta = \arccos\left(\frac{\int_0^1 (1+t)(1+t^2)\,dt}{\sqrt{\int_0^1 (1+t)^2\,dt}\sqrt{\int_0^1 (1+t^2)^2\,dt}}\right) = \arccos\left(\frac{25/12}{\sqrt{7/3}\sqrt{28/15}}\right) \approx .059 \text{ radians.}$$

   (b) $v_1(t) = 1$, $v_2(t) = \sqrt{3}(2t - 1)$, and $v_3(t) = \sqrt{5}\left(6t^2 - 6t + 1\right)$.

   (c) $1 + t^2 = \frac{4}{3}\ 1 + \frac{1}{2\sqrt{3}}\ \sqrt{3}(2t - 1) + \frac{1}{6\sqrt{5}}\ \sqrt{5}\left(6t^2 - 6t + 1\right)$.

3. (a) Let $\mathbf{v} \in V_0$ and let α be a nonzero scalar. Then $\langle \alpha\mathbf{v}, \mathbf{p} \rangle = \bar{\alpha}\langle \mathbf{v}, \mathbf{p} \rangle = 0 \Rightarrow \alpha\mathbf{v} \in V_0$. Suppose $\mathbf{v}, \mathbf{w} \in V_0$. Then $\langle \mathbf{v} + \mathbf{w}, \mathbf{p} \rangle = \langle \mathbf{v}, \mathbf{p} \rangle + \langle \mathbf{w}, \mathbf{p} \rangle = 0 \Rightarrow \mathbf{v} + \mathbf{w} \in V_0$. Therefore, V_0 is a subspace since it is closed under scalar multiplication and vector addition.

   (b) Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for V and assume that $\langle \mathbf{p}, \mathbf{u}_1 \rangle \neq 0$. (This must be true for some basis vector since $\mathbf{p} \in V$, and we are thus assuming without loss of generality that B is ordered so that $\mathbf{p}$ is not orthogonal to the first basis vector.)

Since B is a basis for V and $\langle \mathbf{p}, \mathbf{u}_1 \rangle \neq 0$, we can use the Gram-Schmidt process starting with the set B—but with $\mathbf{p}$ replacing $\mathbf{u}_1$—to produce an orthonormal basis B' for V: $B' = \left\{ \dfrac{\mathbf{p}}{\|\mathbf{p}\|}, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1} \right\}$, where the $\mathbf{v}_i$ are the vectors generated by the Gram-Schmidt procedure. Claim: the set $B_0 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}\}$ is a basis for V_0. This immediately implies that $\dim(V_0) = n - 1$. To prove the claim, we need to show that B_0 is linearly independent and spans V_0. The linear independence follows from the fact that B_0 is, by construction, an orthonormal set. To show that B_0 spans V_0, consider $\mathbf{w} \in V_0$. Since $\mathbf{w}$ is also in V and B' is an orthonormal basis for V, the Fourier expansion of $\mathbf{w}$ with respect to B' is

$$\begin{aligned}
\mathbf{w} &= \left\langle \mathbf{w}, \frac{\mathbf{p}}{\|\mathbf{p}\|} \right\rangle \frac{\mathbf{p}}{\|\mathbf{p}\|} + \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_{n-1} \rangle \mathbf{v}_{n-1} \\
&= \langle \mathbf{w}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{w}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{w}, \mathbf{v}_{n-1} \rangle \mathbf{v}_{n-1}
\end{aligned}$$

because $\langle \mathbf{w}, \mathbf{p}/\|\mathbf{p}\| \rangle = \dfrac{1}{\|\mathbf{p}\|} \langle \mathbf{w}, \mathbf{p} \rangle = 0$. Thus, every $\mathbf{w} \in V_0$ can be written as a combination of vectors in $B_0 \Rightarrow V_0 = \text{span}(B_0)$.