Key to Problem Set #5

1. (a) The 4 expected outputs are $(c\ c\ c\ c)^{\mathrm{T}} = (1\ 1\ 1\ 1)^{\mathrm{T}}\,(c) = \mathbf{Ax}$. Equating these to the 4 actual outputs $(b_1\ b_2\ b_3\ b_4)^{\mathrm{T}} = \mathbf{b}$ gives the linear system $\mathbf{Ax} = \mathbf{b}$.

   (b) $\mathbf{P} = \mathbf{A}(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}(4)^{-1}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix}$.

   (c) $\bar{c} = \left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{b} = \frac{1}{4}\left(b_1 + b_2 + b_3 + b_4\right)$. That is, the best least-squares estimate of c is the simple average of the observations.

2. Let n be the size of $\mathbf{I}_n$. If $n = 1$, $\mathbf{I}_1 = (1)$ $\det(\mathbf{I}_1) = 1$ by definition of the determinant of a 1×1 matrix. Now assume $\det(\mathbf{I}_n) = 1$ and consider $\det(\mathbf{I}_{n+1})$. Expanding along the first row of $\mathbf{I}_{n+1}$ gives $\det(\mathbf{I}_{n+1}) = [\mathbf{I}_{n+1}]_{11}(-1)^{(1+1)}\det(\mathbf{M}_{11}) + [\mathbf{I}_{n+1}]_{12}(-1)^{(1+2)}\det(\mathbf{M}_{12}) + \quad + [\mathbf{I}_{n+1}]_{1(n+1)}(-1)^{(1+(n+1))}\det(\mathbf{M}_{1(n+1)}) = 1\ \det(\mathbf{M}_{11})$ since the first element in row 1 of $\mathbf{I}_{n+1}$ is unity and is the only nonzero element in that row. But $\mathbf{M}_{11}$ is formed by deleting the first row and first column of $\mathbf{I}_{n+1}$ $\quad \mathbf{M}_{11} = \mathbf{I}_n$ $\quad \det(\mathbf{M}_{11}) = \det(\mathbf{I}_n) = 1$ $\quad \det(\mathbf{I}_{n+1}) = 1\ 1 = 1$. Thus, by induction, the determinant of any identity matrix equals one.

3. $(1 - t^2)^3$

4. (a) $0 = \det(\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}) = \det([\mathbf{A}^{\mathrm{T}} - \lambda\mathbf{I}]^{\mathrm{T}}) = \det(\mathbf{A} - \lambda\mathbf{I})$. Thus $\lambda \quad \sigma\left(\mathbf{A}^{\mathrm{T}}\right)$ iff $\lambda \quad \sigma(\mathbf{A})$.

   (b) $\mathbf{y}^*\mathbf{A} = \mu\mathbf{y}^* \Leftrightarrow \mathbf{y}^*\left(\mathbf{A} - \mu\mathbf{I}\right) = \mathbf{0} \Leftrightarrow \mathbf{0} = \left(\mathbf{A} - \mu\mathbf{I}\right)^*\mathbf{y} = \left(\mathbf{A}^* - \bar{\mu}\mathbf{I}\right)\mathbf{y} = \left(\mathbf{A}^{\mathrm{T}} - \bar{\mu}\mathbf{I}\right)\mathbf{y}$ since $\mathbf{A}$ is real. Thus, μ is a "left-hand" eigenvalue of $\mathbf{A} \Leftrightarrow \bar{\mu} \quad \sigma\left(\mathbf{A}^{\mathrm{T}}\right) \Leftrightarrow \bar{\mu} \quad \sigma(\mathbf{A})$ (using part a) $\Leftrightarrow \mu \quad \sigma(\mathbf{A})$ (using the hint).

   (c) Consider $\langle \mathbf{y}, \mathbf{Ax}\rangle = \mathbf{y}^*(\mathbf{Ax}) = \left(\mathbf{y}^*\mathbf{A}\right)\mathbf{x} = \lambda_1\mathbf{y}^*\mathbf{x} = \lambda_1\langle \mathbf{y}, \mathbf{x}\rangle$. But we also have $\langle \mathbf{y}, \mathbf{Ax}\rangle = \langle \mathbf{y}, \lambda_2\mathbf{x}\rangle = \lambda_2\langle \mathbf{y}, \mathbf{x}\rangle$. Thus $\langle \mathbf{y}, \mathbf{Ax}\rangle = \lambda_1\langle \mathbf{y}, \mathbf{x}\rangle = \lambda_2\langle \mathbf{y}, \mathbf{x}\rangle \quad \left(\lambda_1 - \lambda_2\right)\langle \mathbf{y}, \mathbf{x}\rangle = 0 \quad \langle \mathbf{y}, \mathbf{x}\rangle = 0 \quad \mathbf{y} \quad \mathbf{x}$.