Homework #2 Partial Key

1. (b) $E(g_i) = q_t \cdot 1 + (1-q_t) \cdot 0 = q_t$

$E(w) = q_t \cdot w_A + (1-q_t)w_a = \bar{w}_t$

$E(gw) = q_t \cdot 1 \cdot w_A + (1-q_t) \cdot 0 \cdot w_a = q_t w_A$

so

$$\begin{aligned}
\text{cov}(g_i, w_i) &= E(g_i w_i) - E(g_i) \cdot E(w_i) \\
&= q_t w_A - q_t \cdot \left[q_t w_A + (1-q_t)w_a\right] \\
&= q_t w_A - q_t^2 w_A - q_t(1-q_t)w_a \\
&= (q_t - q_t^2)w_A - q_t(1-q_t)w_a \\
&= q_t(1-q_t)w_A - q_t(1-q_t)w_a \\
&= q_t(1-q_t)(w_A - w_a)
\end{aligned}$$

(c)

$$\begin{aligned}
\Delta\bar{w}_t = \bar{w}_{t+1} - \bar{w}_t &= \left[q_{t+1}w_A + (1-q_{t+1})w_a\right] - \left[q_t w_A + (1-q_t)w_a\right] \\
&= (q_{t+1} - q_t)w_A + \left[(1-q_{t+1}) - (1-q_t)\right]w_a = (q_{t+1} - q_t)w_A + \left[q_t - q_{t+1}\right]w_a \\
&= \Delta q_t w_A - \Delta q_t w_a \\
&= \Delta q_t (w_A - w_a)
\end{aligned}$$

Substituting in $\Delta q_t = q_t(1-q_t)(w_A - w_a)/\bar{w}_t$ gives

$$\Delta\bar{w}_t = \frac{q_t(1-q_t)(w_A - w_a)}{\bar{w}_t} \cdot (w_A - w_a) = \frac{q_t(1-q_t)(w_A - w_a)^2}{\bar{w}_t}$$

(d)

$$\begin{aligned}
\text{var}(w_i) &= q_t w_A^2 + (1-q_t)w_a^2 - \left[q_t w_A + (1-q_t)w_a\right]^2 \\
&= q_t w_A^2 + (1-q_t)w_a^2 - q_t^2 w_A^2 - 2q_t(1-q_t)w_A w_a - (1-q_t)^2 w_a^2 \\
&= (q_t - q_t^2)w_A^2 - q_t(1-q_t)2w_A w_a + \left[(1-q_t) - (1-q_t)^2\right]w_a^2 \\
&= q_t(1-q_t)w_A^2 - q_t(1-q_t)2w_A w_a - q_t(1-q_t)w_a^2 \\
&= q_t(1-q_t)(w_A^2 - 2w_A w_a - w_a^2) \\
&= q_t(1-q_t)(w_A - w_a)^2
\end{aligned}$$

3. (b)

$$\frac{d\bar{w}}{d\bar{z}} = \frac{d}{d\bar{z}}\int w(z)\phi(z)dz$$
$$= \int w(z)\frac{d}{d\bar{z}}\phi(z)dz$$
$$= \int w(z)\frac{d}{d\bar{z}}\left\{\frac{1}{\sqrt{2\pi V_z}}\exp\left[-\frac{(z-\bar{z})^2}{2V_z}\right]\right\}dz$$
$$= \int w(z)\left\{\frac{(z-\bar{z})}{V_z}\frac{1}{\sqrt{2\pi V_z}}\exp\left[-\frac{(z-\bar{z})^2}{2V_z}\right]\right\}dz$$
$$= \int w(z)\frac{(z-\bar{z})}{V_z}\phi(z)dz$$
$$= \frac{1}{V_z}\left[\int w(z)z\phi(z)dz - \bar{z}\int w(z)\phi(z)dz\right]$$
$$= \frac{1}{V_z}\left[\int w(z)z\phi(z)dz - \bar{z}\bar{w}\right]$$
$$= \frac{\operatorname{cov}(w,z)}{V_z}$$
$$= \frac{\bar{w}S}{V_z}$$

Since $\beta = \frac{S}{V_z}$, this shows that $\beta = \frac{1}{\bar{w}}\frac{d\bar{w}}{d\bar{z}}$.